Feedback – An Example, Stability, Stationary Errors

Automatic Control, Basic Course, Lecture 4

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Feedback – The Steam Engine

Control in the Old Days

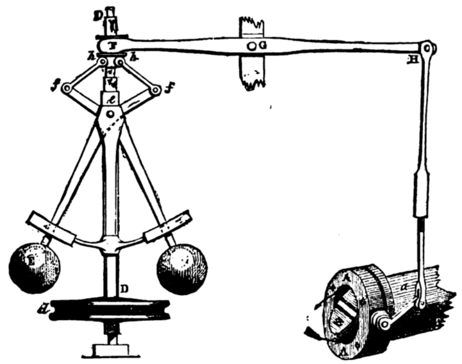
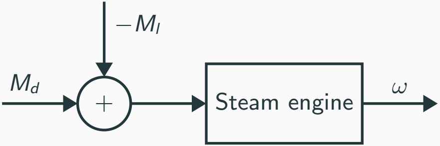


FIG. 4.—*Governor and Throttle-Valve.*

The Uncontrolled Steam Engine



Model:

$$J\dot{\omega} + D\omega = M_d - M_l$$

The stationary angular speed:

$$\omega_s = \frac{M_d - M_l}{D}$$

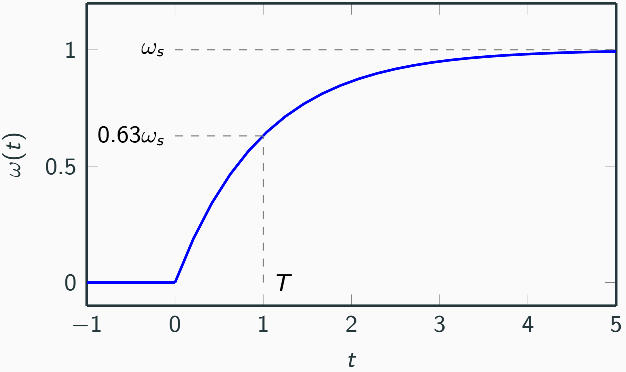
The Uncontrolled Steam Engine

Step response:

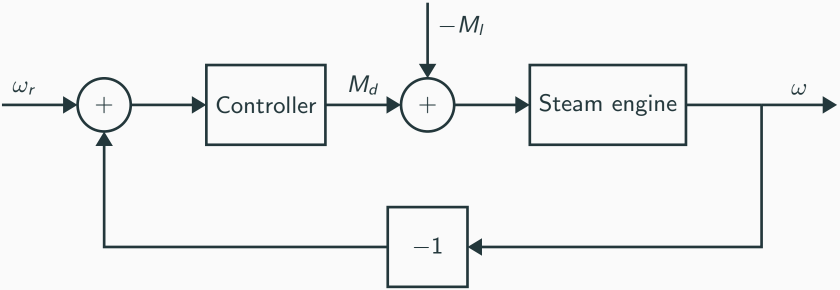
$$\omega(t) = \frac{M_d - M_l}{D}\left(1 - e^{-Dt/J}\right) = \omega_s\left(1 - e^{-Dt/J}\right)$$

Time constant:

$$T = \frac{J}{D}$$



P Control of the Steam Engine



Proportional control:

$$M_d = K(\omega_r - \omega)$$

P Control of the Steam Engine

Dynamics with P-controller:

$$J\dot{\omega} + D\omega = \overbrace{K(\omega_r - \omega)}^{M_d} - M_l$$

or

$$J\dot{\omega} + (D + K)\omega = K\omega_r - M_l$$

In stationarity ($\dot{\omega} = 0$):

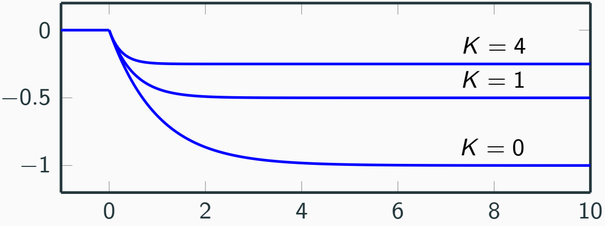
$$\omega_s = \frac{K}{D+K}\omega_r - \frac{1}{D+K}M_l$$

Step response ($\omega(0) = 0$):

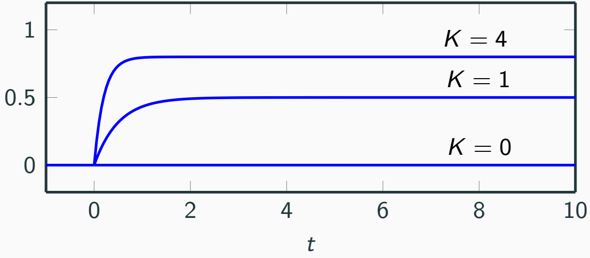
$$\omega(t) = \frac{K\omega_r - M_b}{D+K}\left(1 - e^{-(D+K)t/J}\right) = \omega_s\left(1 - e^{-(D+K)t/J}\right)$$

P Control of the Steam Engine

Angular speed $\omega(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):



Driving torque $M_d(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):



PI Control of the Steam Engine

Introduce a PI-controller to get rid of the stationary error:

$$M_d = K(\omega_r - \omega) + \frac{K}{T_i}\int_0^t (\omega_r - \omega)\mathrm{d}t$$

Dynamics:

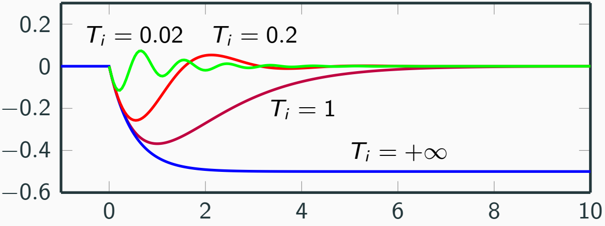
$$\begin{aligned} J\dot{\omega} + D\omega &= K(\omega_r - \omega) + \frac{K}{T_i}\int_0^t (\omega_r - \omega)\mathrm{d}t - M_l \\ J\ddot{\omega} + D\dot{\omega} &= K(\dot{\omega}_r - \dot{\omega}) + \frac{K}{T_i}(\omega_r - \omega) - \dot{M}_l \end{aligned}$$

At stationarity ($\dot{\omega}_r = 0$, $\dot{M}_l = 0$):

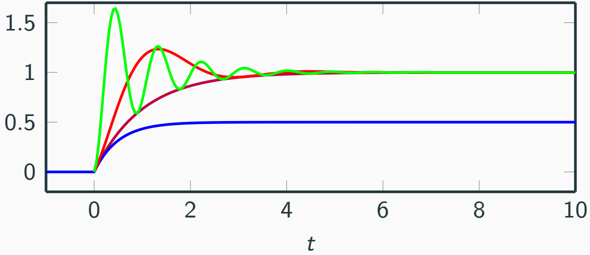
$$\omega_s = \omega_r$$

PI Control of the Steam Engine

Angular speed $\omega(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):



Driving torque $M_d(t)$ ($\omega_r = 0$ and $M_l = \theta(t)$):



PI Control of the Steam Engine

The Laplace transformation of the dynamics

$$J\ddot{\omega} + D\dot{\omega} = K(\dot{\omega}_r - \dot{\omega}) + \frac{K}{T_i}(\omega_r - \omega) - \dot{M}_l$$

is

$$s^2 J\omega + sD\omega = K(s\omega_r - s\omega) + \frac{K}{T_i}(\omega_r - \omega) - sM_l$$

The characteristic equation (the equation to determine the poles) is:

$$s^2 + \frac{D + K}{J}s + \frac{K}{J\,T_i} = 0$$

By choosing K and T_i, we can place the poles of the closed loop dynamics arbitrarily.

Stability

Stability - Definitions

A system on state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is

Asymptotically stable if $x(t) \to 0$ when $t \to +\infty$ for all initial states $x(0)$ when $u(t) = 0$.

Stable if $x(t)$ is bounded for all t and all initial states $x(0)$ when $u(t) = 0$.

Unstable if $x(t)$ grows unbounded for an initial state $x(0)$ when $u(t) = 0$.

Stability - Scalar Case

For the scalar case

$$\dot{x}(t) = ax(t)$$
$$x(0) = x_0$$

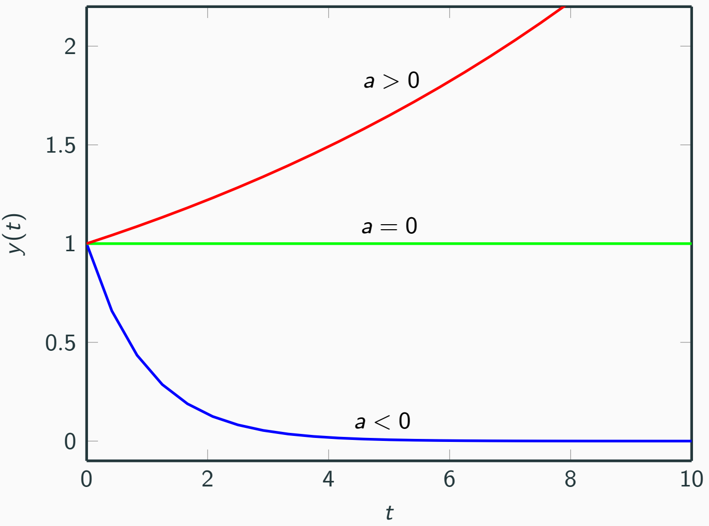
the solution is:

$$x(t) = x_0 e^{at}$$

Hence

$a < 0$	Asymptotically stable
$a = 0$	(marginally) Stable
$a > 0$	Unstable

Stability - Scalar Case



Stability - Diagonal Case

$$\dot{x}(t) = \overbrace{\begin{bmatrix} a_1 & & & 0 \\ & a_2 & & \\ & & \ddots & \\ 0 & & & a_n \end{bmatrix}}^{A} x(t)$$
$$x(0) = x_0$$

Every state variable corresponds to the scalar case:

$$\dot{x}_i(t) = a_i x_i(t)$$

In fact the a_i's are eigenvalues of A. The system is

Asymptotically stable if all the eigenvalues of A have negative real part

Unstable if at least one of the eigenvalues of A has a positive real part

(marginally) Stable if all the eigenvalues of A have either negative or zero real part

Stability - General Case

For a general A-matrix, i.e., not necessarily a diagonal one, the stability rule still holds with one exception. That the eigenvalues have zero real part does not always guarantee stability, unless the purely imaginary eigenvalues are unique.

Stability - Transfer Function

Recall from Lecture 2 that the eigenvalues of the A matrix are poles to the transfer function. Hence, if all the poles have negative real part the system is stable.

A second order polynomial

$$s^2 + a_1 s + a_2$$

has its roots in the left half plane if and only if $a_1 > 0$ and $a_2 > 0$.

A third order polinomial

$$s^3 + a_1 s^2 + a_2 s + a_3$$

has its roots in the left half plane if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and

$$a_1 a_2 > a_3$$

Stability - Example

Example

Determine if the systems below are asymptotically stable or not

a)

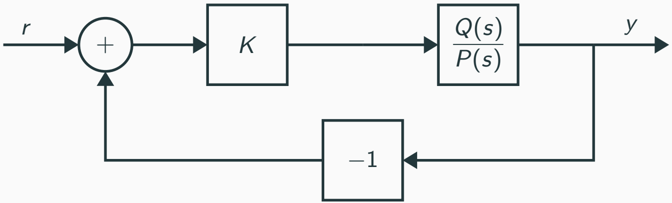
$$G(s) = \frac{1}{(s^2 + s + 1)(s + 1)}$$

b)

$$\dot{x} = \begin{bmatrix} -2 & 2 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 2 & -1 \end{bmatrix} x + 2u$$

Root Locus

Idea: Study graphically how the poles move with the change of a parameter



$$Y(s) = \frac{KQ(s)}{P(s) + KQ(s)} R(s)$$

Characteristic equation:

$$P(s) + KQ(s) = 0$$

Root Locus

Characteristic equation:

$$P(s) + KQ(s) = 0$$

For $K = 0$ the characteristic equation becomes:

$$P(s) = 0$$

When $K \to \infty$, the characteristic equation becomes:

$$Q(s) = 0$$

I.e., the poles of the closed loop system will approach the zeros of the closed loop system.

If there are more poles than zeros, the remaining poles will approach infinity (in magnitude).

Root Locus - Second Order System

Let

$$\frac{Q(s)}{P(s)} = \frac{1}{s(s+1)}$$

Characteristic equation of the cloosed loop:

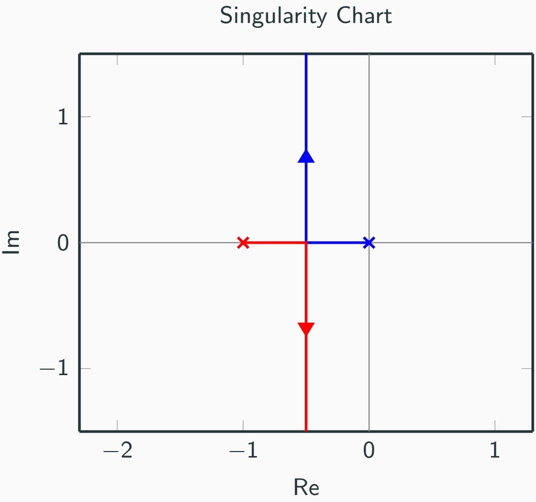
$$P(s) + KQ(s) = s(s+1) + K = 0$$

$$s = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - K}$$

When $K = 0$, poles in 0, -1.

When $K > 1/4$, complex pair of poles with real part $-1/2$. The imaginary parts go towards $\pm\infty$ when $K \to \infty$.

Root Locus - Second Order System



Root Locus - Third Order System

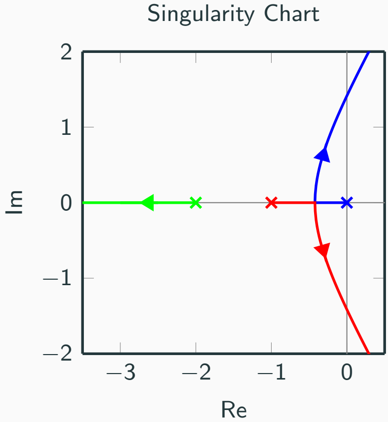
Let

$$\frac{Q(s)}{P(s)} = \frac{1}{s(s+1)(s+2)}$$

Characteristic equation of the closed loop:

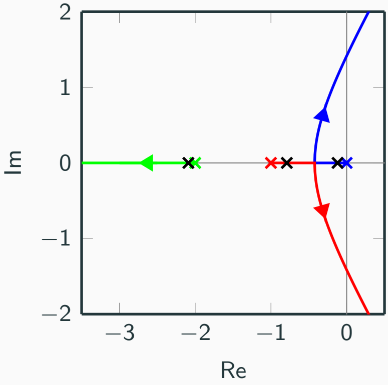
$$P(s) + KQ(s) = s(s+1)(s+2) + K = s^3 + 3s^2 + 2s + K = 0$$

Root Locus - Third Order System

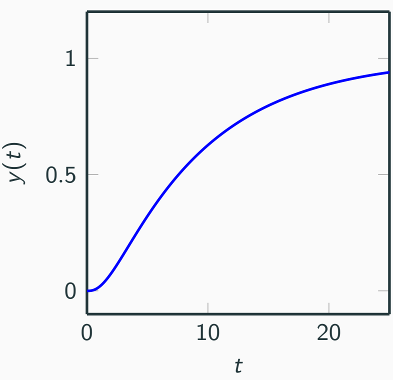


Root Locus - Third Order System

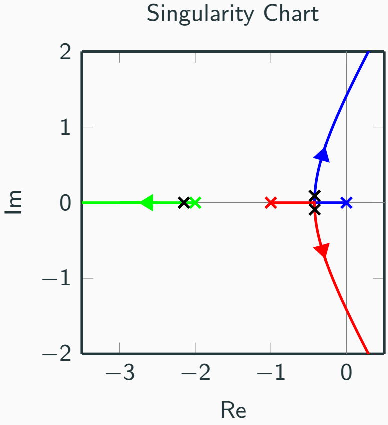
Singularity Chart

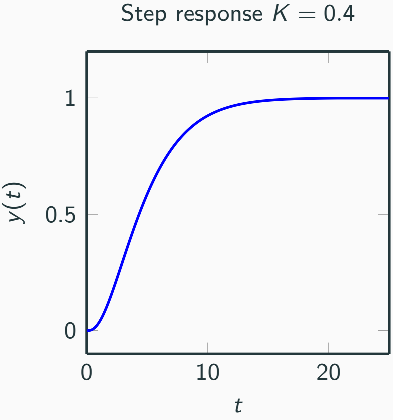


Step response $K = 0.2$

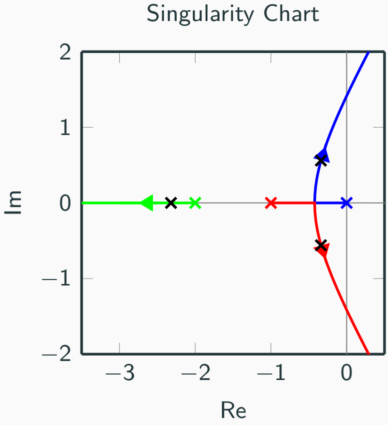


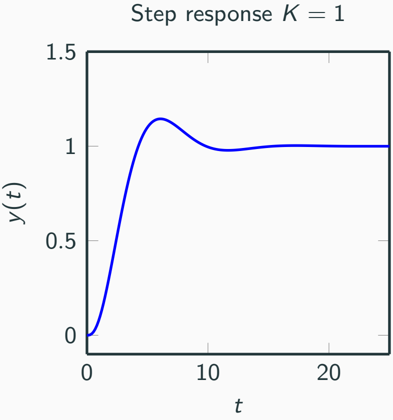
Root Locus - Third Order System



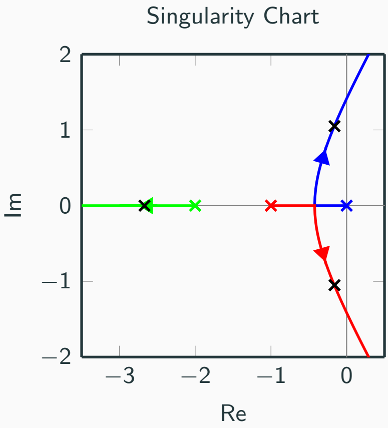


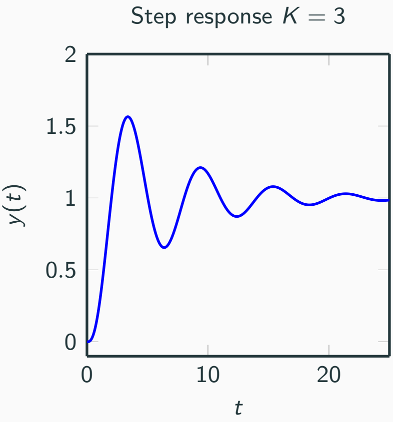
Root Locus - Third Order System



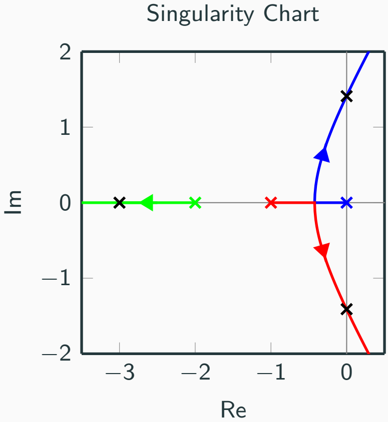


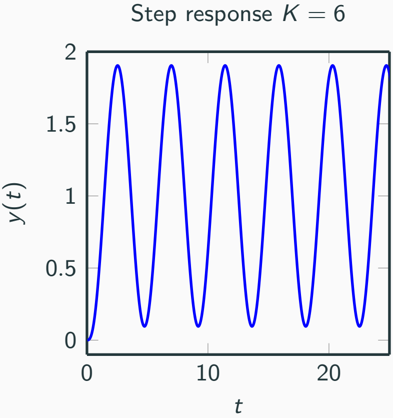
Root Locus - Third Order System



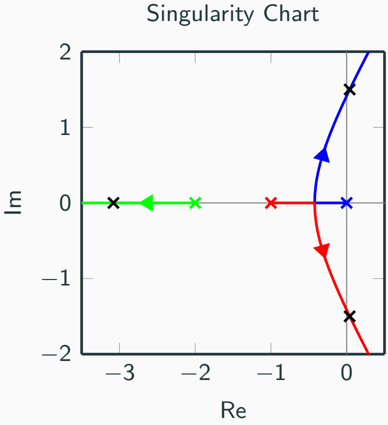


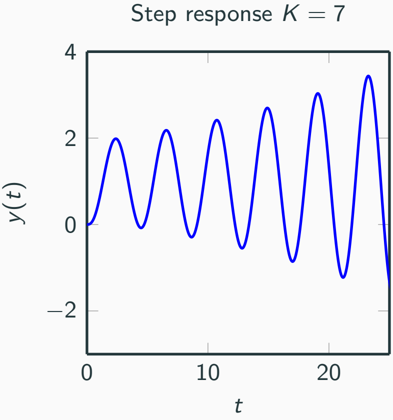
Root Locus - Third Order System





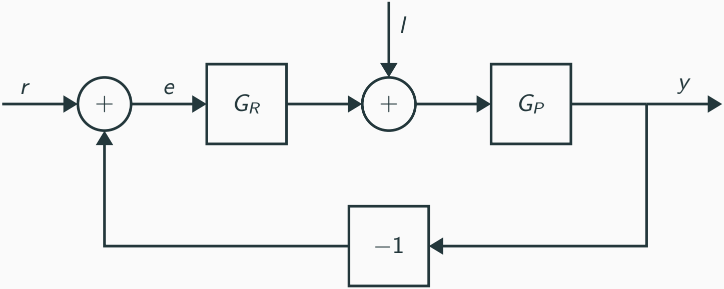
Root Locus - Third Order System





Stationary Errors

The Servo Problem and The Regulator Problem



$$Y = \frac{G_R G_P}{1 + G_R G_P} R + \frac{G_P}{1 + G_R G_P} L$$

The Servo Problem Set point tracking, $l = 0$.

The Regulator Problem Effect of load disturbances, $r = 0$.

Stationary Errors - The Servo Problem

$$E(s) = R(s) - Y(s) = \frac{1}{1 + \underbrace{G_R(s)G_P(s)}_{G_0(s)}} R(s)$$

We can use the final value theorem to determine the error

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s)$$

but only if $sE(s)$ has it poles in the left half plane.

Stationary Errors - The Servo Problem - Example

Let the process and controller be:

$$G_P = \frac{1}{s(1+sT)} \quad G_R = K$$

Open-loop transfer function:

$$G_0 = G_R G_P = \frac{K}{s(s+sT)}$$

The control error is given by:

$$E(s) = \frac{1}{1+G_0(s)} R(s) = \frac{s(1+sT)}{s(1+sT)+K} R(s)$$

Stationary Errors - The Servo Problem - Example

The control error is given by:

$$E(s) = \frac{1}{1 + G_0(s)} R(s) = \frac{s(1 + sT)}{s(1 + sT) + K} R(s)$$

Let $r(t)$ be a step, i.e.,

$$r(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad R(s) = \frac{1}{s}$$

Then (given that T and K are positive)

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \cdot \frac{s(1 + sT)}{s(1 + sT) + K} \cdot \frac{1}{s} = 0$$

Stationary Errors - The Servo Problem - Example

The control error is given by:

$$E(s) = \frac{1}{1 + G_0(s)} R(s) = \frac{s(1 + sT)}{s(1 + sT) + K} R(s)$$

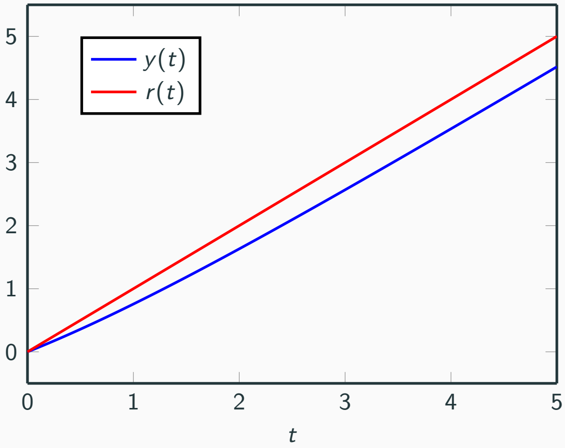
Let $r(t)$ be a ramp, i.e.,

$$r(t) = \begin{cases} t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \quad R(s) = \frac{1}{s^2}$$

Then (given that T and K are positive)

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \cdot \frac{s(1 + sT)}{s(1 + sT) + K} \cdot \frac{1}{s^2} = \frac{1}{K}$$

Stationary Errors - The Servo Problem - Example



Question to the audience: What value of K is used?

Stationary Errors - The Servo Problem - General Case

Open loop transfer function:

$$G_0(s) = \frac{K}{s^n} \cdot \frac{1 + b_1 s + b_2 s^2 + \dots}{1 + a_1 s + a_2 s^2 + \dots} e^{-sL} = \frac{KB(s)}{s^n A(s)} e^{-sL}$$

Setpoint (m non-negative integer):

$$r(t) = \begin{cases} \frac{t^m}{m!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} \qquad R(s) = \frac{1}{s^{m+1}}$$

Error (given that the limit exsists):

$$e_\infty = \lim_{t \to +\infty} e(t) = \lim_{s \to 0} \frac{s^n A(s)}{s^n A(s) + KB(s) e^{-sL}} \cdot \frac{1}{s^{m+1}} = \lim_{s \to 0} \frac{1}{s^n + K} s^{n-m}$$

The stationary error is determined by the low-frequency properties of the transfer function and the set point.

Stationary Errors - The Servo Problem - General Case

$$G_0(s) = \frac{K}{s^n} \cdot \frac{1 + b_1 s + b_2 s^2 + \dots}{1 + a_1 s + a_2 s^2 + \dots} e^{-sL} = \frac{KB(s)}{s^n A(s)} e^{-sL}$$

$$r(t) = \begin{cases} \frac{t^m}{m!} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The relation between m and n gives the following errors:

$$\begin{array}{cl} n > m & e_\infty = 0 \\ n = m = 0 & e_\infty = \frac{1}{1+K} \\ n = m \geq 1 & e_\infty = \frac{1}{K} \\ n < m & \text{Limit does not exist.} \end{array}$$

Stationary Errors - The Regulator Problem

The transfer function between a load disturbance $l(t)$ and measurement signal $y(t)$:

$$Y(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)} L(s)$$

Since $r = 0$, we can study the measurement signal instead of the error:

$$y_\infty = \lim_{t \to +\infty} y(t) = \lim_{s \to 0} sY(s)$$

Again, we have to ensure that the limit exsists.

Stationary Errors - The Regulator Problem - Example

Let the process and controller be:

$$G_P = \frac{1}{1+sT} \quad G_R = \frac{K}{s}$$

Let the load disturbance $l(t)$ be a step:

$$l(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The final theorem yields:

$$y_\infty = \lim_{t \to +\infty} y(t) = \lim_{s \to 0} \frac{s}{s(1+sT)+K} = 0$$

Stationary Errors - The Regulator Problem - Example

Let the process and controller instead be:

$$G_P = \frac{1}{s(1+sT)} \quad G_R = K$$

Notice that $G_0 = G_P G_R$ is the same as in the previous slide. Let the load disturbance $l(t)$ be a step:

$$l(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

The final theorem yields:

$$y_\infty = \lim_{t \to +\infty} y(t) = \lim_{s \to 0} \frac{1}{s(1+sT)+K} = \frac{1}{K}$$

In the regulator problem, the placement of integrators matters

Stationary Errors - The Regulator Problem - General Case

Let

$$G_P(s) = \frac{K_P B_P(s)}{s^p A_p(s)} e^{-sL} \quad G_R(s) = \frac{K B_R(s)}{s^r A_R(s)}$$

where $A_P(0) = B_P(0) = A_R(0) = B_R(0) = 1$. Moreover, let the load disturbances be given by

$$L(s) = \frac{1}{s^{m+1}}$$

Then

$$y_\infty = \lim_{s \to 0} \frac{K_P}{s^{r+p} + K K_P} s^{r-m}$$

The stationary becomes (given that the limits exists):

$$\begin{array}{cl} r > m & y_\infty = 0 \\ r = m = 0,\ p = 0 & y_\infty = \frac{K_P}{1 + K K_P} \\ r = m = 0,\ p \geq 0 & y_\infty = \frac{1}{K} \\ r = m \geq 1 & y_\infty = \frac{1}{K} \\ r < m & \text{The limit does not exist.} \end{array}$$

Stationary Errors - Example

Example

The transfer function of a process is

$$G_p(s) = \frac{1}{s+1}.$$

The process is controlled with a PI-regulator,

$$G_r(s) = 1 + \frac{2}{s}.$$

The closed loop system is able to follow step changes in the reference value without any stationary error, but when the reference is a ramp-signal, $r(t) = ct$, a stationary error emerges. Determine the magnitude of this stationary error.