

Step Response Analysis. Frequency Response, Relation Between Model Descriptions

Automatic Control, Basic Course, Lecture 3

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17 November 2016

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1. Step Response Analysis
2. Frequency Response
3. Relation between Model Descriptions

Step Response Analysis

Step Response

From the last lecture, we know that if the input $u(t)$ is a step, then the output in the Laplace domain is

$$Y(s) = G(s)U(s) = G(s)\frac{1}{s}$$

It is possible to do an inverse transform of $Y(s)$ to get $y(t)$, but is it possible to claim things about $y(t)$ by only studying $Y(s)$?

We will study how the poles affects the step response. (The zeros will be discussed later)

Initial and Final Value Theorem

Let $F(s)$ be the Laplace transformation of $f(t)$, i.e., $F(s) = \mathcal{L}(f(t))(s)$.
Given that the limits below exist, it holds that:

$$\text{Initial value theorem} \quad \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow +\infty} sF(s)$$

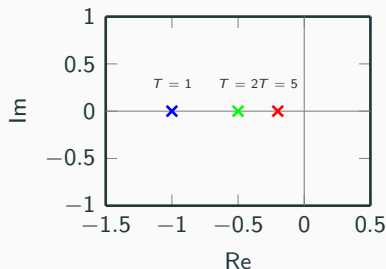
$$\text{Final value theorem} \quad \lim_{t \rightarrow +\infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

For a step response we have that:

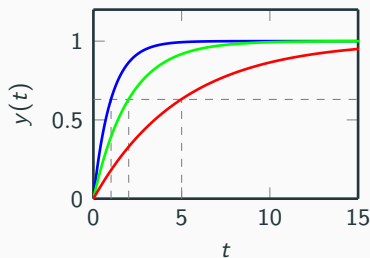
$$\lim_{t \rightarrow +\infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} sG(s) \frac{1}{s} = G(0)$$

First Order System

Singularity Chart



Step Response



$$G(s) = \frac{K}{1 + sT}$$

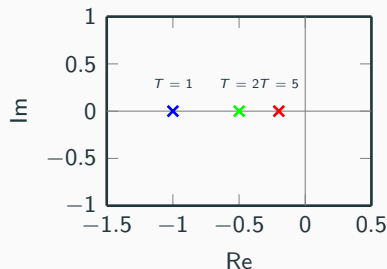
One pole in $s = -1/T$

Step response:

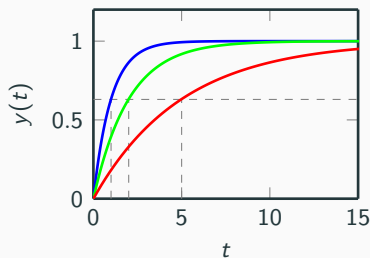
$$Y(s) = G(s) \frac{1}{s} = \frac{K}{s(1 + sT)} \xrightarrow{\mathcal{L}^{-1}} y(t) = K \left(1 - e^{-t/T} \right)$$

First Order System

Singularity Chart



Step Response

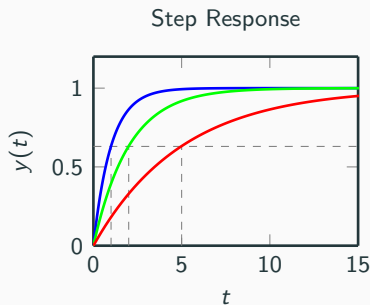
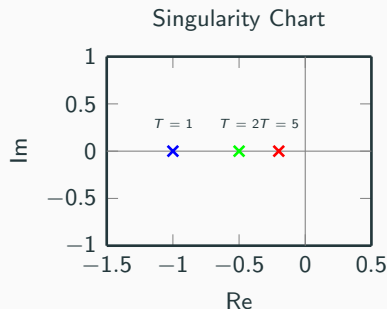


$$G(s) = \frac{K}{1 + sT}$$

Final value:

$$\lim_{t \rightarrow +\infty} y(t) = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{K}{s(1 + sT)} = K$$

First Order System



$$G(s) = \frac{K}{1 + sT}$$

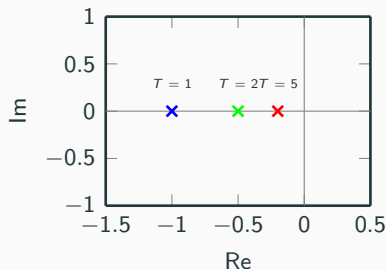
T is called the time-constant:

$$y(T) = K(1 - e^{-T/T}) = K(1 - e^{-1}) \approx 0.63K$$

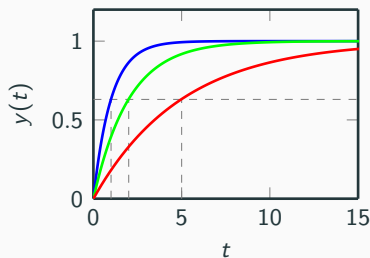
I.e., T is time it takes for the step response to reach 63% of its final value

First Order System

Singularity Chart



Step Response

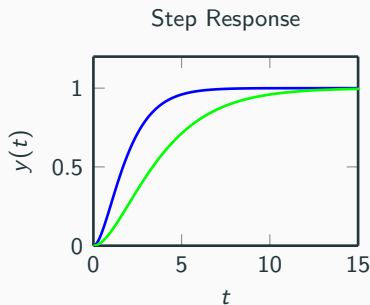
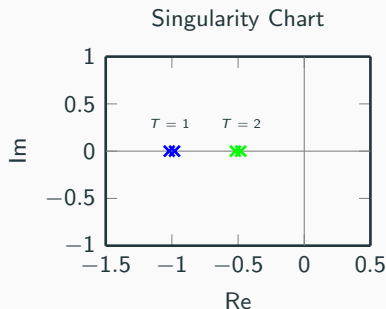


$$G(s) = \frac{K}{1 + sT}$$

Derivative at zero:

$$\lim_{t \rightarrow 0} \dot{y}(t) = \lim_{s \rightarrow +\infty} s \cdot sY(s) = \lim_{s \rightarrow +\infty} \frac{s^2 K}{s(1 + sT)} = \frac{K}{T}$$

Second Order System With Real Poles

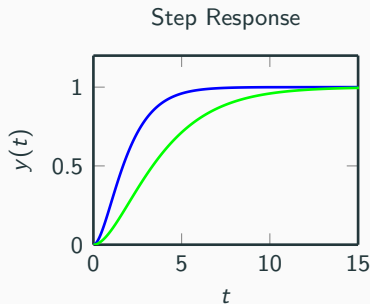
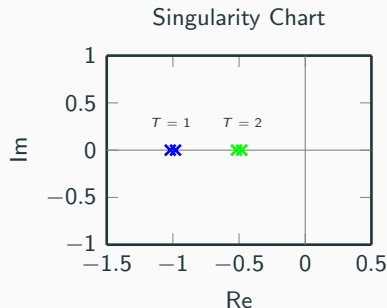


$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Poles in $s = -1/T_1$ and $s = -1/T_2$. Step response:

$$y(t) = \begin{cases} K \left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2} \right) & T_1 \neq T_2 \\ K \left(1 - e^{-t/T} - \frac{t}{T} e^{-t/T} \right) & T_1 = T_2 = T \end{cases}$$

Second Order System With Real Poles

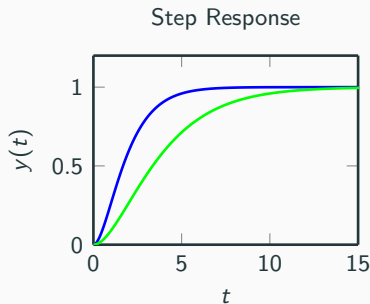
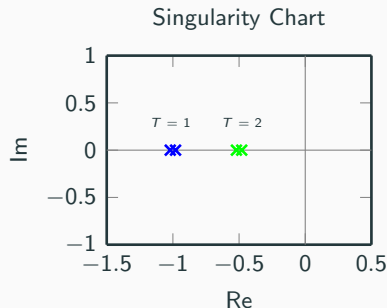


$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Final value:

$$\lim_{t \rightarrow +\infty} = \lim_{s \rightarrow 0} sY(s) = \lim_{s \rightarrow 0} \frac{sK}{s(1 + sT_1)(1 + sT_2)} = K$$

Second Order System With Real Poles



$$G(s) = \frac{K}{(1 + sT_1)(1 + sT_2)}$$

Derivative at zero:

$$\lim_{t \rightarrow 0} \dot{y}(t) = \lim_{s \rightarrow +\infty} s \cdot sY(s) = \lim_{s \rightarrow +\infty} \frac{s^2 K}{s(1 + sT_1)(1 + sT_2)} = 0$$

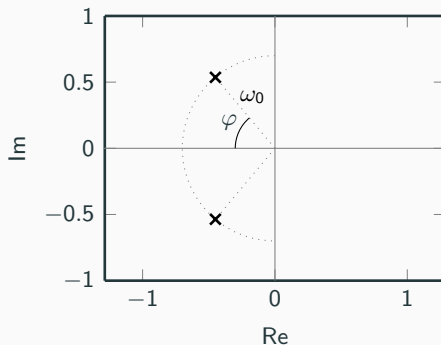
Second Order System With Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \quad 0 < \zeta < 1$$

Relative damping ζ , related to the angle φ

$$\zeta = \cos(\varphi)$$

Singularity Chart



Second Order System With Complex Poles

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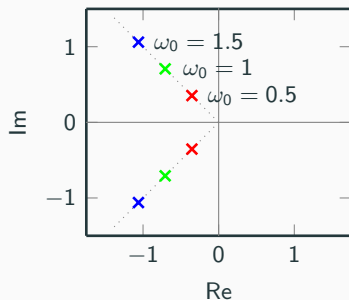
Inverse transformation yields:

$$y(t) = K \left(1 - \frac{1}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_0 t} \sin \left(\omega_0 \sqrt{1 - \zeta^2} t + \arccos \zeta \right) \right)$$

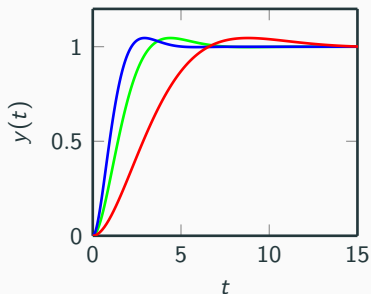
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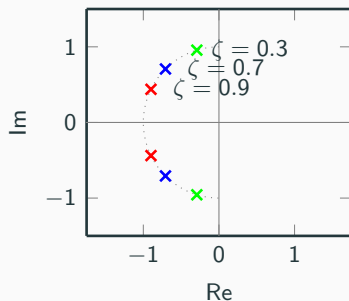
Step Response



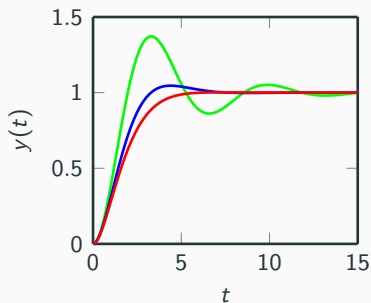
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Singularity Chart



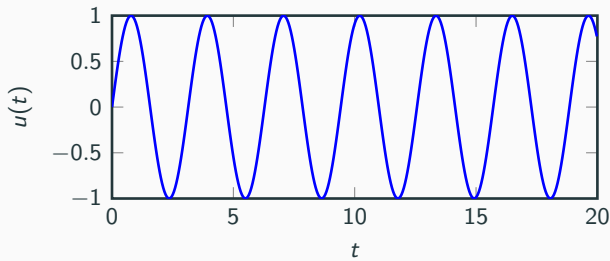
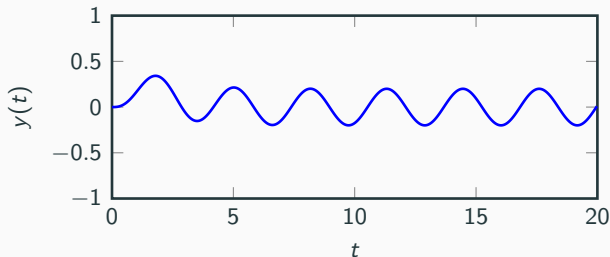
Step Response



Frequency Response

Sinusoidal Input

Given a transfer function $G(s)$, what happens if we let the input be $u(t) = \sin(\omega t)$?



Sinusoidal Input

It can be shown that if the input is $u(t) = \sin(\omega t)$, the output will be

$$y(t) = a \sin(\omega t + \varphi)$$

where

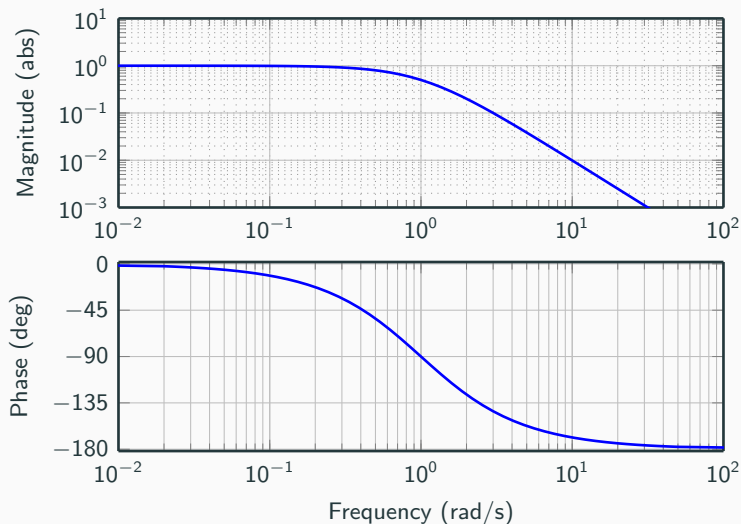
$$a = |G(i\omega)|$$

$$\varphi = \arg G(i\omega)$$

So if we determine a and φ for different frequencies, we have a description of the transfer function.

Bode Plot

Idea: Plot $|G(i\omega)|$ and $\arg G(i\omega)$ for different frequencies ω .



Bode Plot - Products of Transfer Functions

Let

$$G(s) = G_1(s)G_2(s)G_3(s)$$

then

$$\log |G(i\omega)| = \log |G_1(i\omega)| + \log |G_2(i\omega)| + \log |G_3(i\omega)|$$

$$\arg G(i\omega) = \arg G_1(i\omega) + \arg G_2(i\omega) + \arg G_3(i\omega)$$

This means that we can construct Bode plots of transfer functions from simple "building blocks" for which we know the Bode plots.

Bode Plot of $G(s) = K$

If

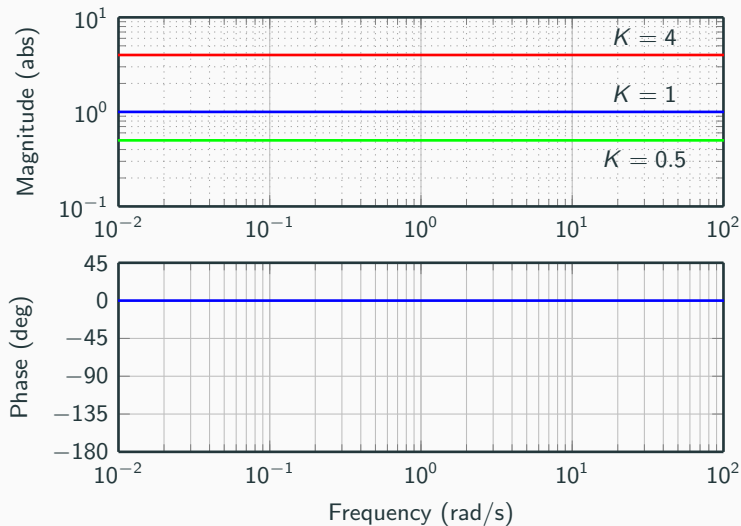
$$G(s) = K$$

then

$$\log |G(i\omega)| = \log(K)$$

$$\arg G(i\omega) = 0$$

Bode Plot of $G(s) = K$



Bode Plot of $G(s) = s^n$

If

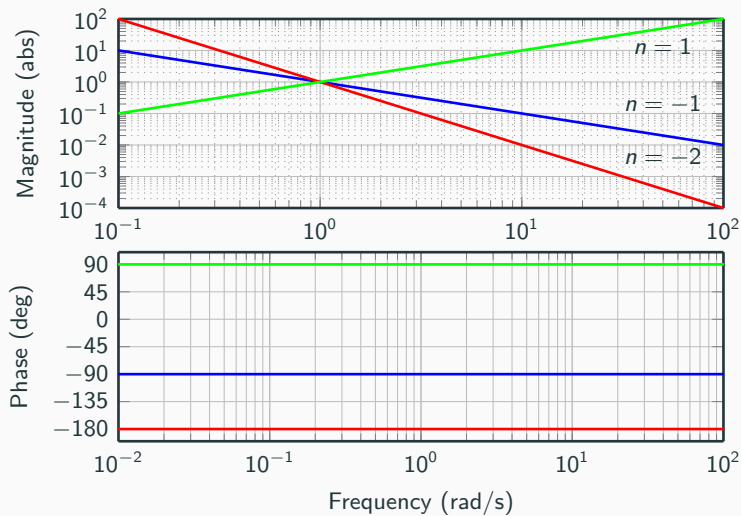
$$G(s) = s^n$$

then

$$\log |G(i\omega)| = n \log(\omega)$$

$$\arg G(i\omega) = n \frac{\pi}{2}$$

Bode Plot of $G(s) = s^n$



Bode Plot of $G(s) = (1 + sT)^n$

If

$$G(s) = (1 + sT)^n$$

then

$$\log |G(i\omega)| = n \log(\sqrt{1 + \omega^2 T^2})$$

$$\arg G(i\omega) = n \arg(1 + i\omega T) = n \arctan(\omega T)$$

For small ω

$$\log |G(i\omega)| \rightarrow 0$$

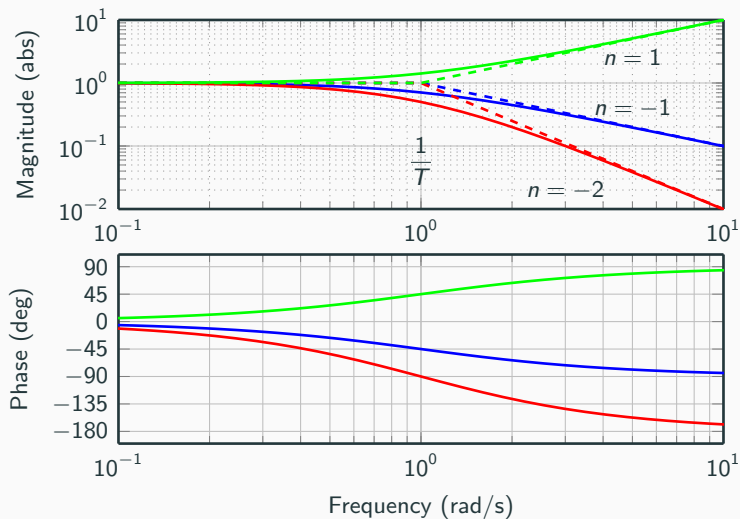
$$\arg G(i\omega) \rightarrow 0$$

For large ω

$$\log |G(i\omega)| \rightarrow n \log(\omega T)$$

$$\arg G(i\omega) \rightarrow n \frac{\pi}{2}$$

Bode Plot of $G(s) = (1 + sT)^n$



Bode Plot of $G(s) = (1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$

$$G(s) = (1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$$

For small ω

$$\log |G(i\omega)| \rightarrow 0$$

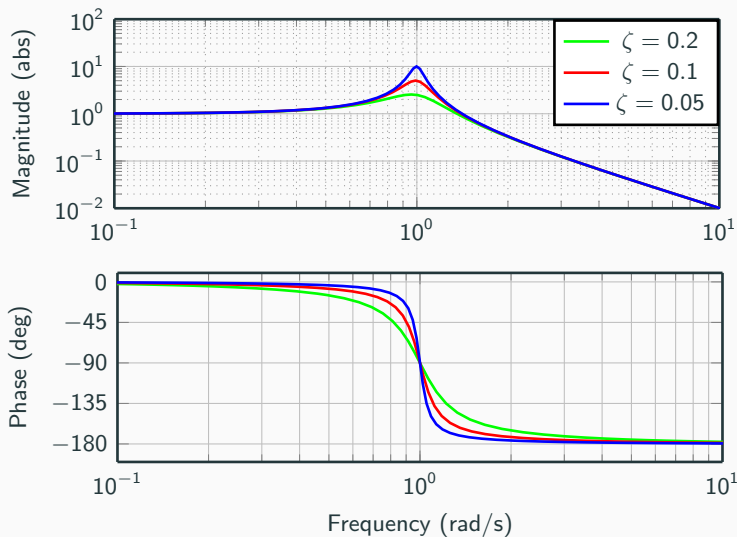
$$\arg(i\omega) \rightarrow 0$$

For large ω

$$\log |G(i\omega)| \rightarrow 2n \log \left(\frac{\omega}{\omega_0} \right)$$

$$\arg G(i\omega) \rightarrow n\pi$$

Bode Plot of $G(s) = (1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$



Bode Plot of $G(s) = e^{-sL}$

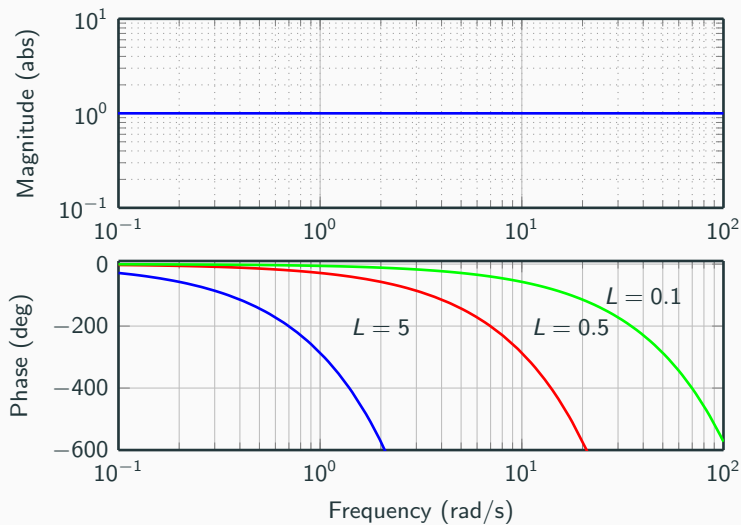
$$G(s) = e^{-sL}$$

Describes a pure time delay with delay L , i.e. $y(t) = u(t - L)$

$$\log |G(i\omega)| = 0$$

$$\arg G(i\omega) = -\omega L$$

Bode Plot of $G(s) = e^{-sL}$



Bode Plot of Composite Transfer Function

Example

Draw the Bode plot of the transfer function

$$G(s) = \frac{100(s + 2)}{s(s + 20)^2}$$

First step, write it as product of sample transfer functions:

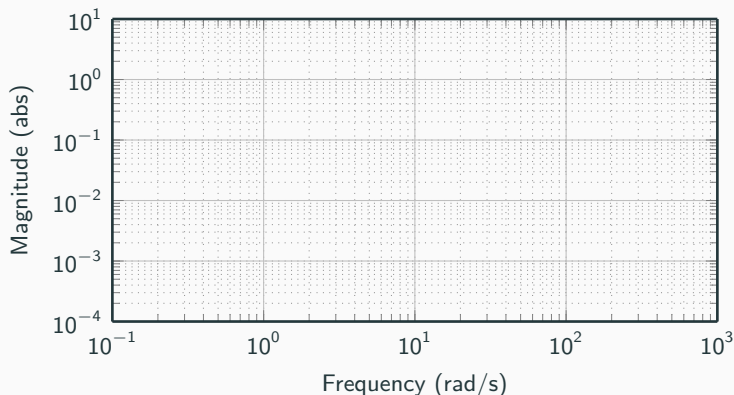
$$G(s) = \frac{100(s + 2)}{s(s + 20)^2} = 0.5 \cdot s^{-1} \cdot (1 + 0.5s) \cdot (1 + 0.05s)^{-2}$$

Then determine the corner frequencies:

$$G(s) = \frac{100(s + 2)}{s(s + 20)^2} = 0.5 \cdot s^{-1} \cdot \overbrace{(1 + 0.5s)}^{w_{c1}=2} \cdot \overbrace{(1 + 0.05s)^{-2}}^{w_{c2}=20}$$

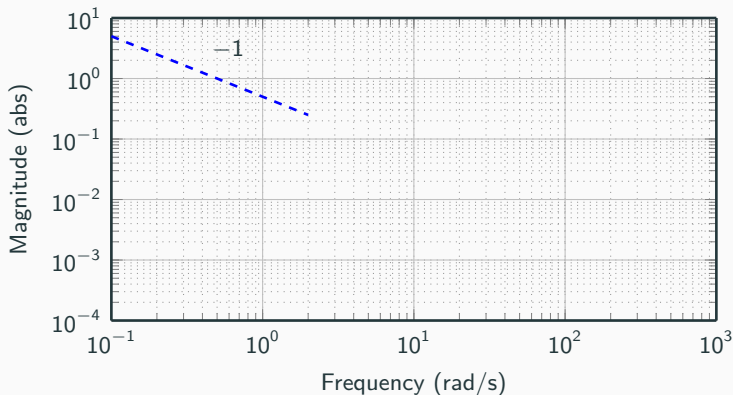
Bode Plot of Composite Transfer Function

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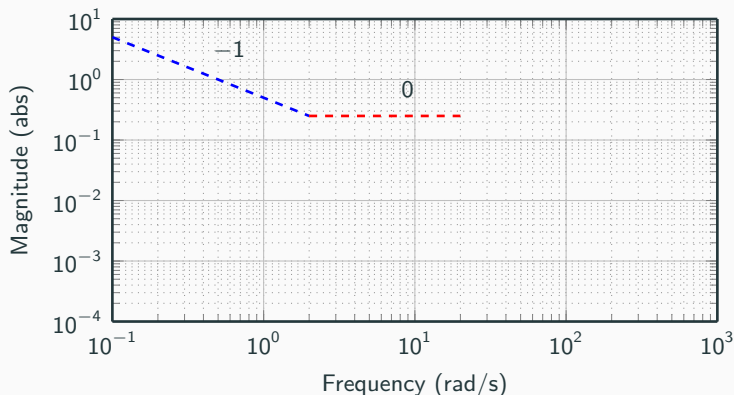
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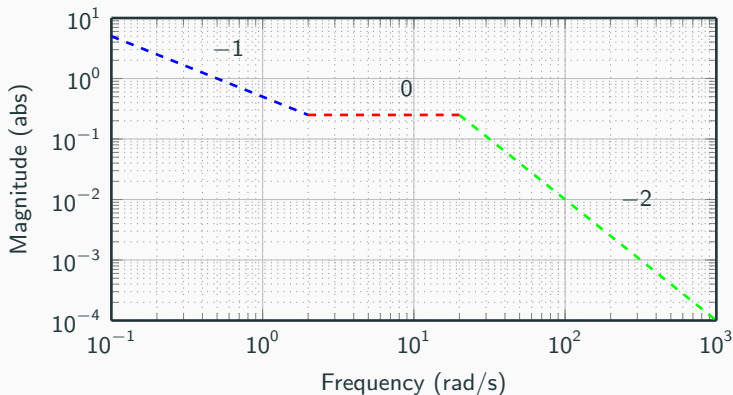
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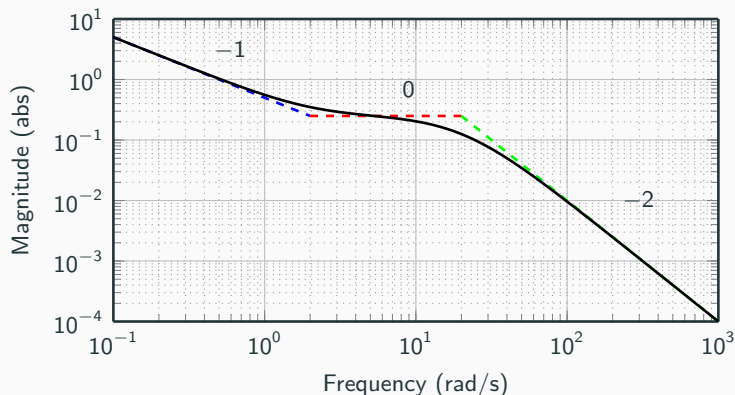
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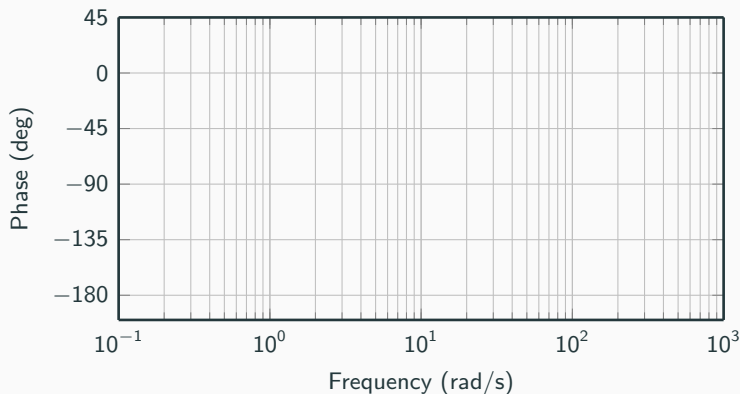
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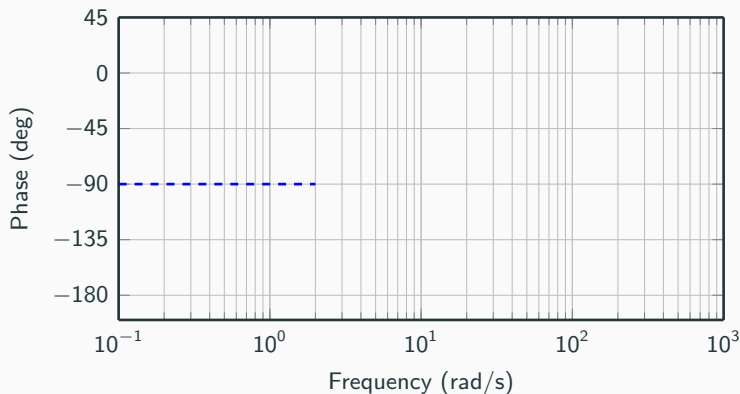
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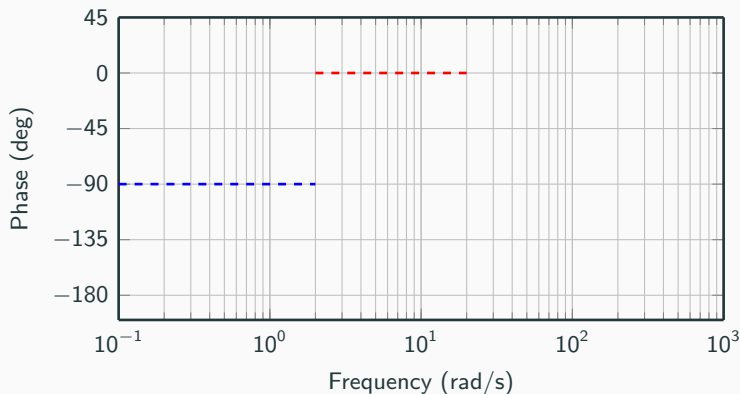
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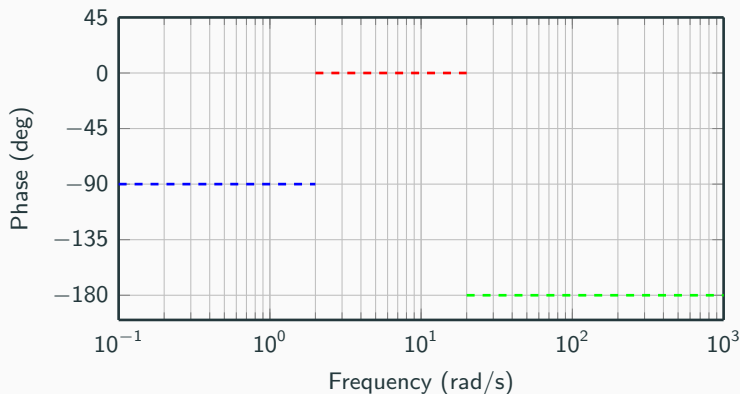
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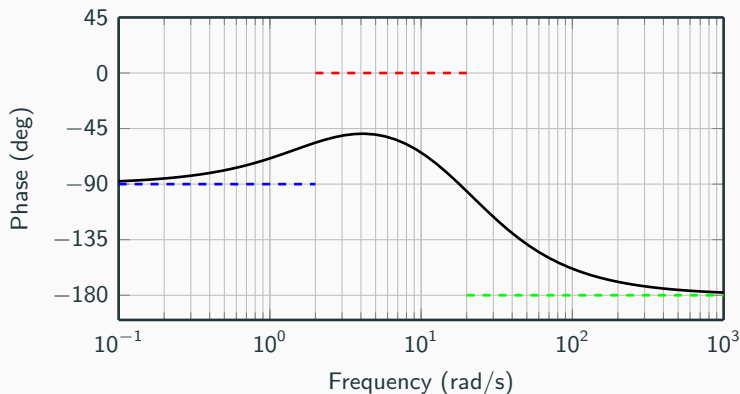
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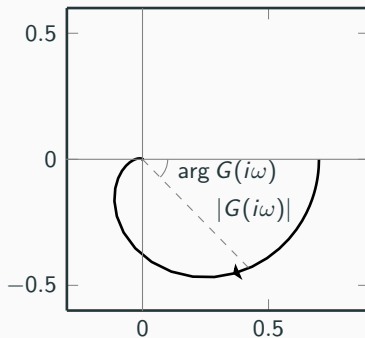
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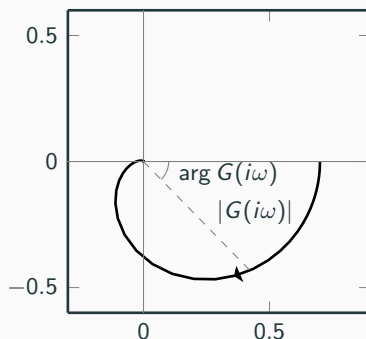
Nyquist Plot

By removing the frequency information, we can plot the transfer function in one plot instead of two



Nyquist Plot

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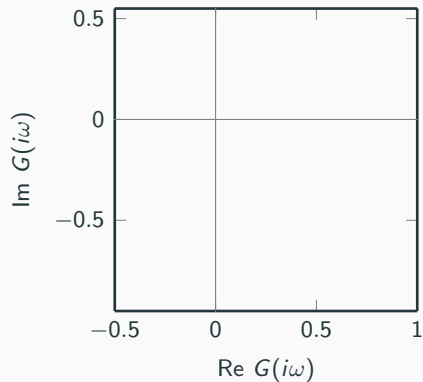
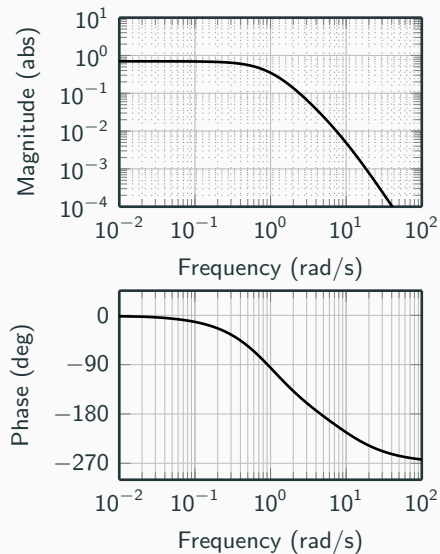


Split the transfer function into real and imaginary part:

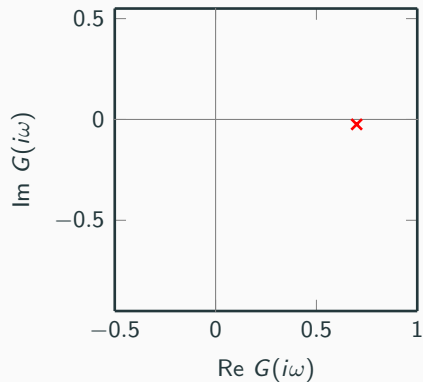
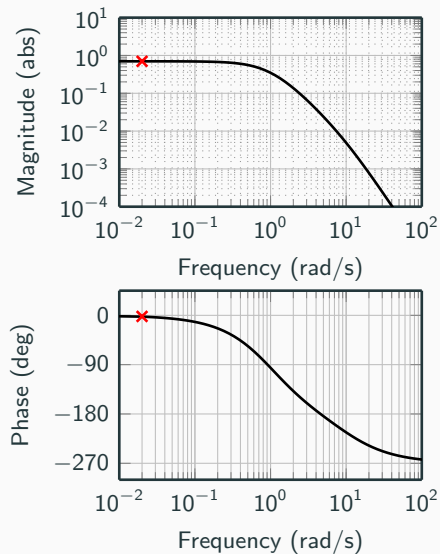
$$G(s) = \frac{1}{1+s} \quad G(i\omega) = \frac{1}{1+i\omega} = \frac{1}{1+\omega^2} - i \frac{\omega}{1+\omega^2}$$

Is this the transfer function in the plot above?

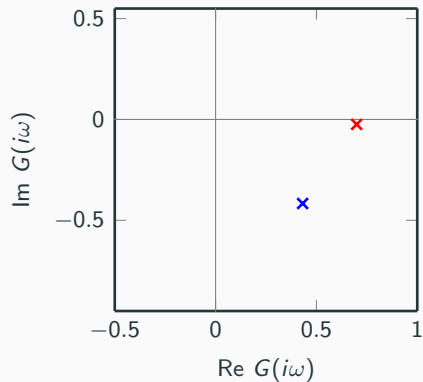
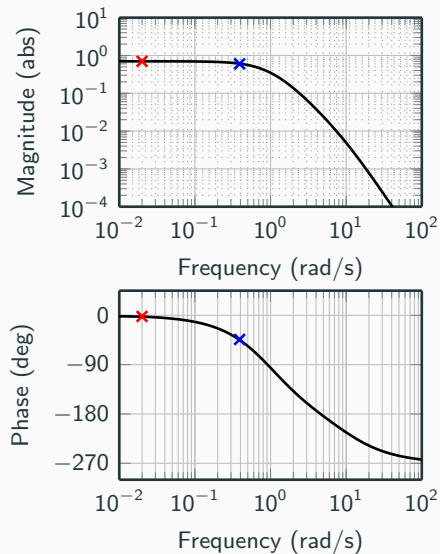
From Bode Plot to Nyquist Plot



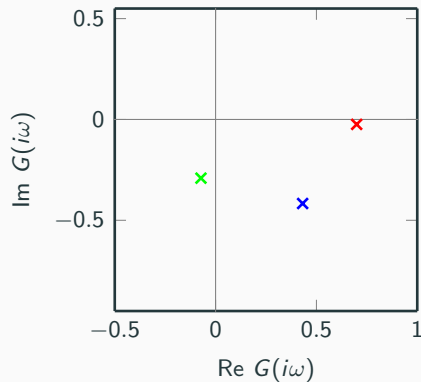
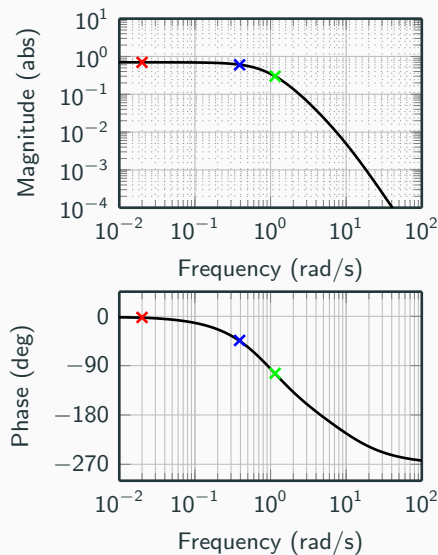
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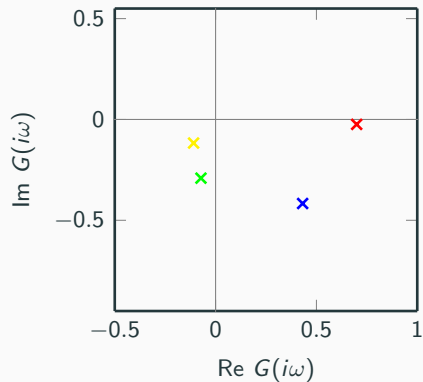
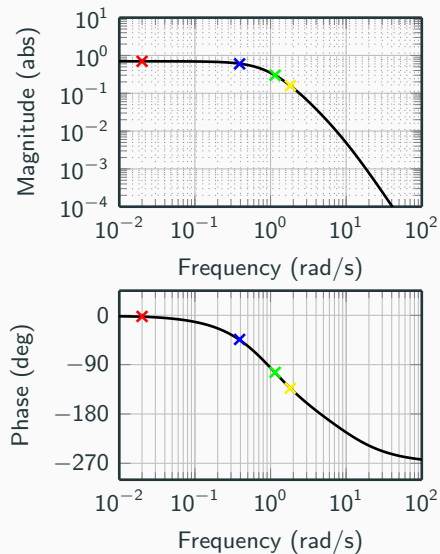
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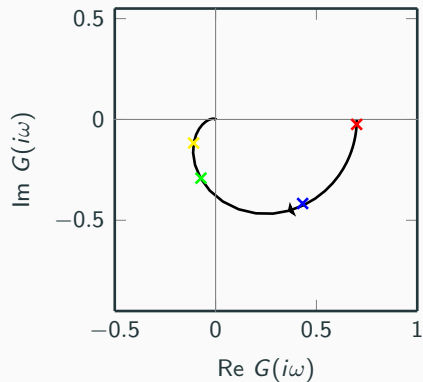
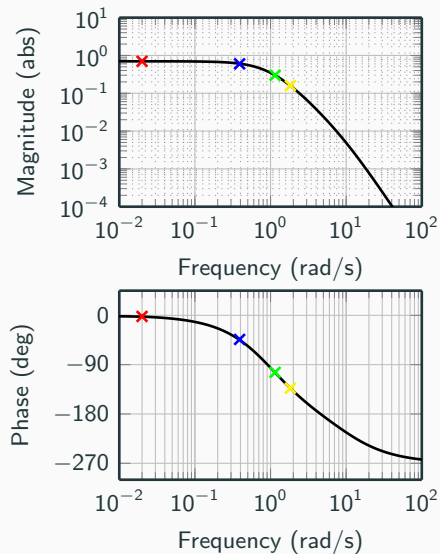
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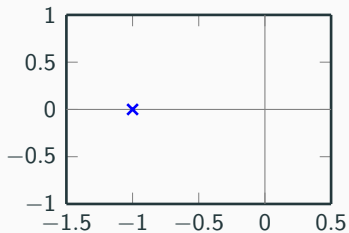
From Bode Plot to Nyquist Plot



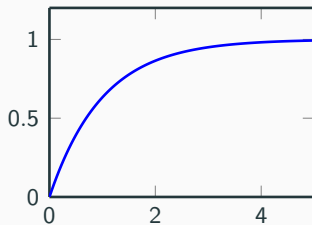
Relation between Model Descriptions

Single-capacitive Processes

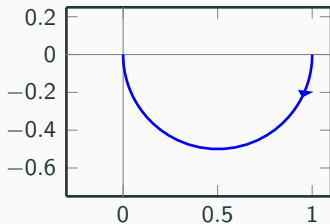
Singularity chart



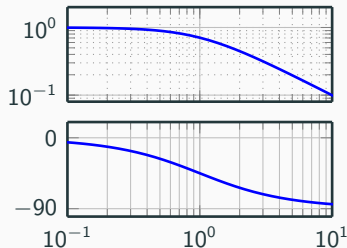
Step response



Nyquist plot

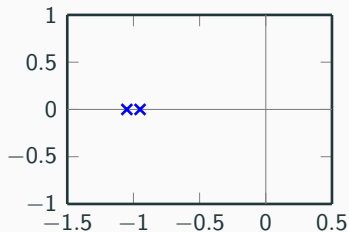


Bode plot

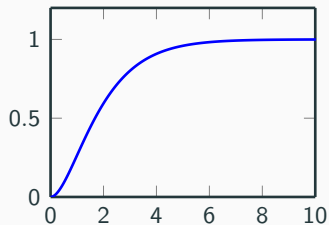


Multi-capacitive Processes

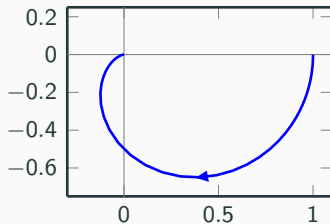
Singularity chart



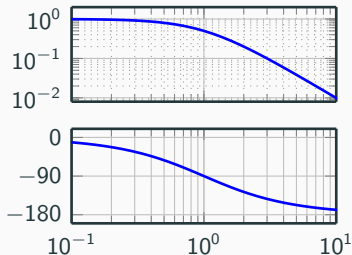
Step response



Nyquist plot

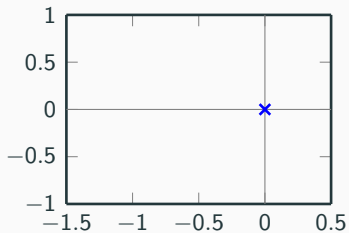


Bode plot

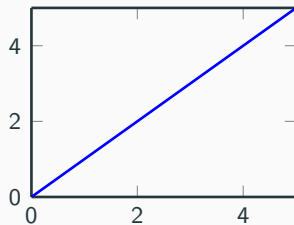


Integrating Processes

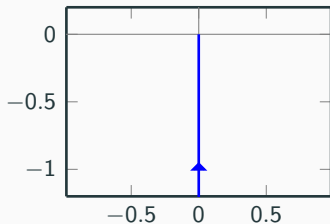
Singularity chart



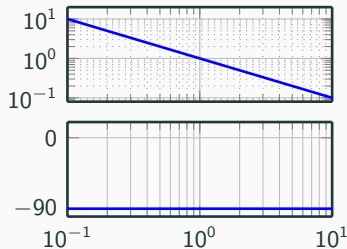
Step response



Nyquist plot

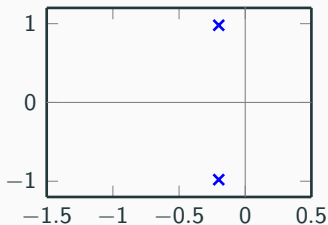


Bode plot

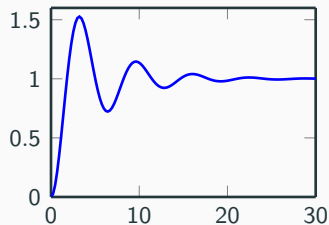


Oscillative Processes

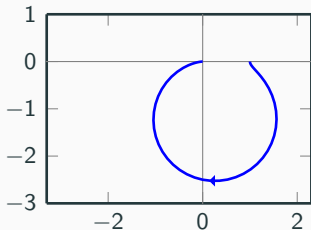
Singularity chart



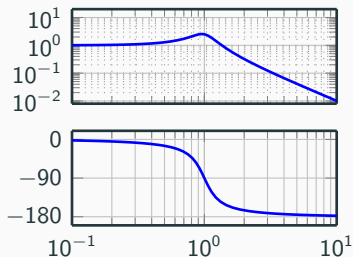
Step response



Nyquist plot

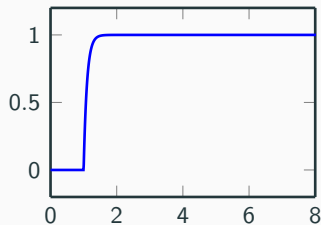


Bode plot

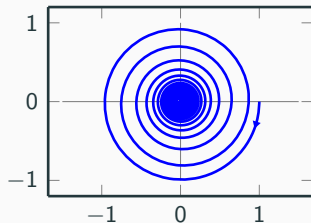


Delay Processes

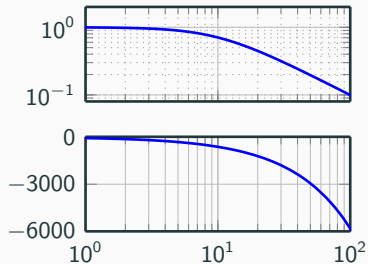
Step response



Nyquist plot

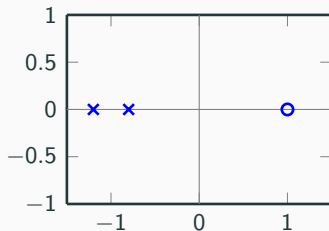


Bode plot

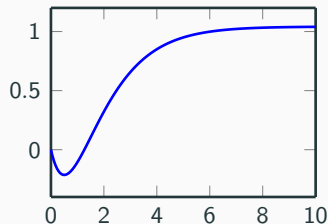


Process with Inverse Responses

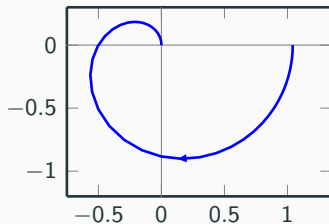
Singularity chart



Step response



Nyquist plot



Bode plot

