Linearization, Transfer Function, Block Diagram Representation, Transient Response

Automatic Control, Basic Course, Lecture 2

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Content

Linearization

Linearization - Why?

Many systems are nonlinear. However, one can approximate them with linear ones. This to get a system that are easier to analyze.

A few examples of nonlinear systems:

- Water tanks (Lab 2)
- Air resistance
- Action potentials in neurons
- Pendulum under the influence of gravity
- ...

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$$f(x, u) \approx f(x_0, u_0) + \frac{\partial}{\partial x} f(x_0, u_0)(x - x_0) + \frac{\partial}{\partial u} f(x_0, u_0)(u - u_0)$$
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3. Introduce $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$
4. The state-space equations in the new variables are given by:

$$\dot{\Delta x} = \dot{x} - \dot{x}_0 = f(x, u) \approx \frac{\partial}{\partial x} f(x_0, u_0)\Delta x + \frac{\partial}{\partial u} f(x_0, u_0)\Delta u = A\Delta x + B\Delta u$$

$$\Delta y = g(x, u) - y_0 \approx \frac{\partial}{\partial x} g(x_0, u_0)\Delta x + \frac{\partial}{\partial u} g(x_0, u_0)\Delta u = C\Delta x + D\Delta u$$

Example - Linearization

Example

The dynamics of a specific system is described by

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1}$$
$$y = x_1^2 + u^2$$

a) Find all stationary points

b) Linearize the system around the stationary point corresponding to $u^0 = 3$

Transfer Function

Laplace Transformation

Let $f(t)$ be a function of time t, the Laplace transformation $\mathcal{L}(f(t))(s)$ is defined as

$$\mathcal{L}(f(t))(s) = F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

Assuming that $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$ (steady assumption during this course) it has the property that

$$\mathcal{L}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right)(s) = s^n F(s)$$

See Collection of Formulae for a table of Laplace transformations.

Example

A system's dynamics is described by the differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_1\dot{u} + b_2 u.$$

After Laplace transformation we get

$$(s^2 + a_1 s + a_2)Y(s) = (b_1 s + b_2)U(s)$$

which can be written as

$$Y(s) = \overbrace{\frac{b_1 s + b_2}{s^2 + a_1 s + a_2}}^{G(s)} U(s) = G(s)U(s)$$

$G(s)$ is called the transfer function of the system.

Transfer Function

Relation between control signal $U(s)$ and output $Y(s)$:

$$Y(s) = G(s)U(s)$$

$G(s)$ often fraction of polynomal, i.e.,

$$G(s) = \frac{Q(s)}{P(s)}$$

Zeros of $Q(s)$ are called zeros of the system, zeros of $P(s)$ are called poles of the system.

The poles play a very important role for the system's behavior.

From State Space to Transfer Function

For a system on state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

the transfer function is given by

$$G(s) = C(sI - A)^{-1}B + D$$

Observe: the denominator of $G(s)$ is given by $P(s) = \det(sI - A)$, so eigenvalues of A are poles of the system.

From Transfer Function to State Space

Can be done in several ways, see Collection of Formulae.

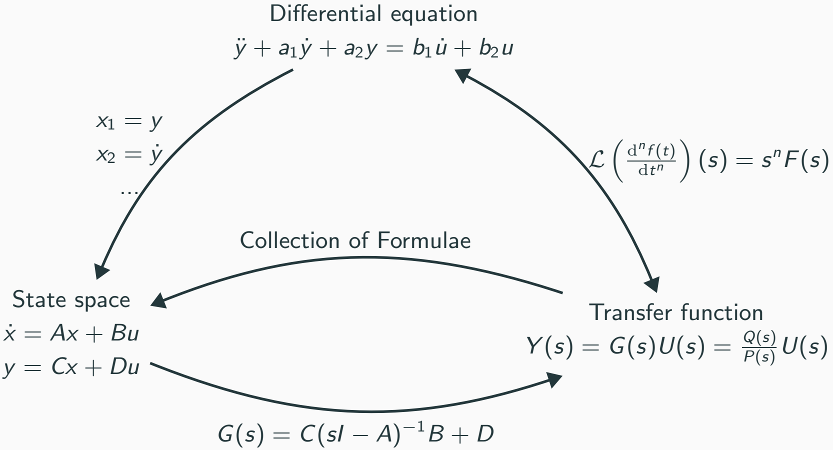
Example

A system's transfer function is

$$G(s) = \frac{2s + 1}{s^3 + 4s - 8}$$

Write the system on a state space form of your choice.

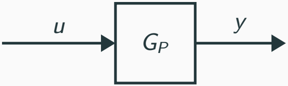
Three Ways to Describe a Dynamical System



Block Diagram Representation

Block Diagram - Transfer Function

When the blocks in a block diagram are replaced by transfer functions, it is possible to describe the relations between signals in a easy way.

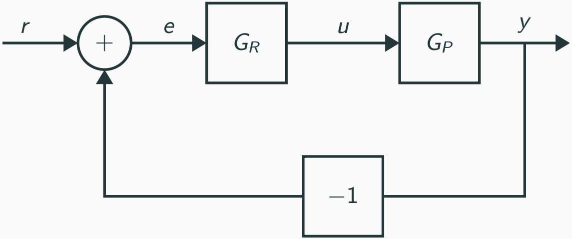


$$Y(s) = G_P(s)U(s)$$

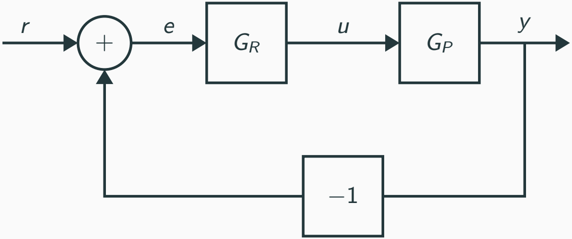
Block Diagram - Components

Most block diagrams consist of three components:

- Blocks - Transfer functions
- Arrows - Signals
- Summations



Determine Transfer Function From Block Diagram



$$Y = G_P U, \quad U = G_R E, \quad E = R - Y$$

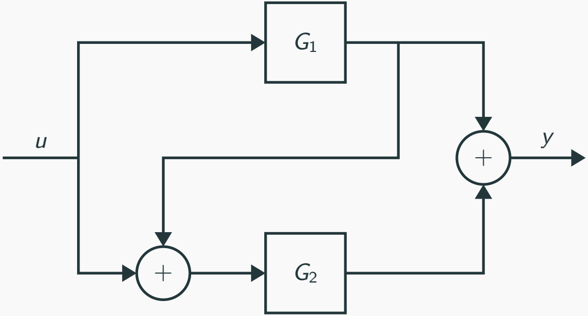
From the equations above the transfer function between r and y is

$$Y = \frac{G_P G_R}{1 + G_P G_R} U$$

Example - Transfer Functions

Example

Two systems, G_1 and G_2, are interconnected as in the figure below



Compute the transfer function from u to y, G_{yu}.

Transient Response

Solution to State Space Equation

Given a system on state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The solution, $y(t)$, is then given by

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

- $Ce^{At}x(0)$: Initial state, uninteresting except when the controller is initialized
- $C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau$: Weighted integral of the control signal, interesting part
- $Du(t)$: Direct term, often neglectable in practical systems

Impulse Response

Shows how the system responds when the input is a short pulse, i.e., a Dirac function

$$u(t) = \delta(t)$$

The Laplace transformation is

$$U(s) = \int_0^\infty e^{-st}\delta(t)\mathrm{d}t = 1$$

Hence

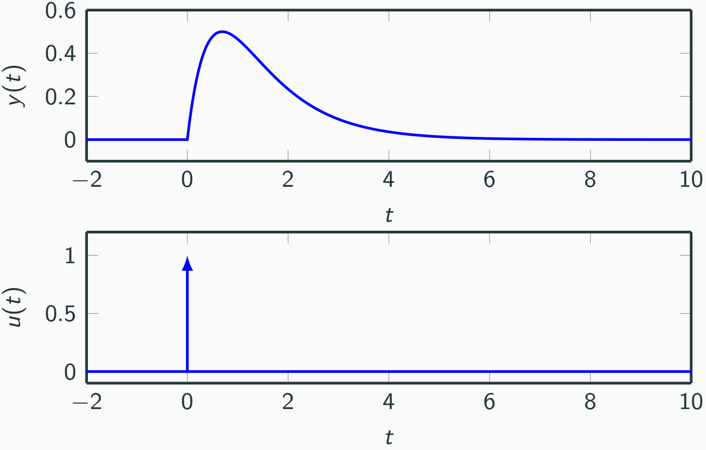
$$Y(s) = G(s)U(s) = G(s)$$

Not so common in technological applications, can we think of other applications?

Example - Impulse Response

Let the transfer function of the system be:

$$G(s) = \frac{2}{s^2 + 3s + 2}$$



Step Response

Shows how the system responds when the input is a step, i.e.,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The Laplace transformation is

$$U(s) = \int_0^\infty e^{-st} u(t) \mathrm{d}t = \int_0^\infty e^{-st} \mathrm{d}t = -\frac{1}{s} \left[e^{-st} \right]_0^\infty = \frac{1}{s}$$

Very common in technological applications

Example - Step Response

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