Lec 5: Frequency Domain Stability Analysis

The Nyquist Criterion. Stability Margins. Sensitivity

November 20, 2017

Lund University, Department of Automatic Control

Stability is Important!

Stability Margins are also Important!

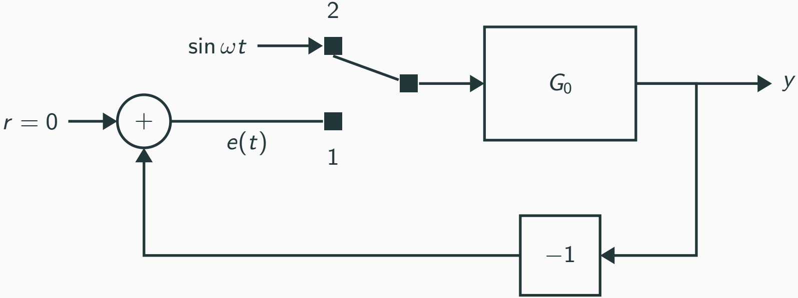
X29

Harry Nyquist (1889-1976)

Nilsby, Sweden → North Dakota → Yale → Bell Labs

- Nyquist's stability criterion
- The Nyquist frequency
- Johnson-Nyquist noise

Nyquist's Criterion — A Motivation



With switch in position 2, after transients (G_0 stable):

$$\begin{aligned} e(t) &= -|G_0(i\omega)| \sin(\omega t + \arg G_0(i\omega)) \\ &= |G_0(i\omega)| \sin(\omega t + \arg G_0(i\omega) + \pi) \end{aligned}$$

Find ω_0 such that $\arg G_0(i\omega_0) = -\pi$.

Also assume $|G_0(i\omega_0)| = 1$

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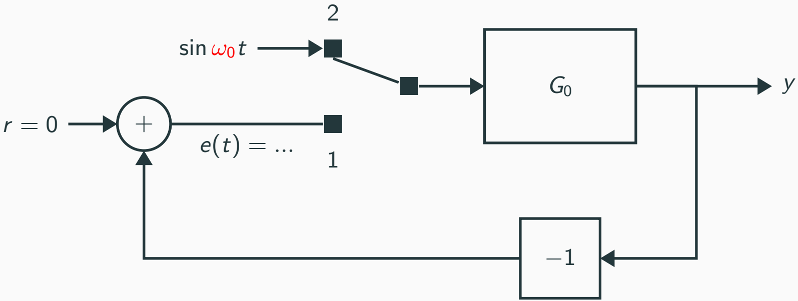

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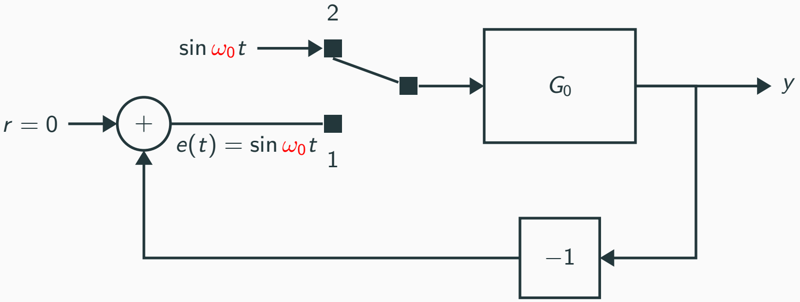

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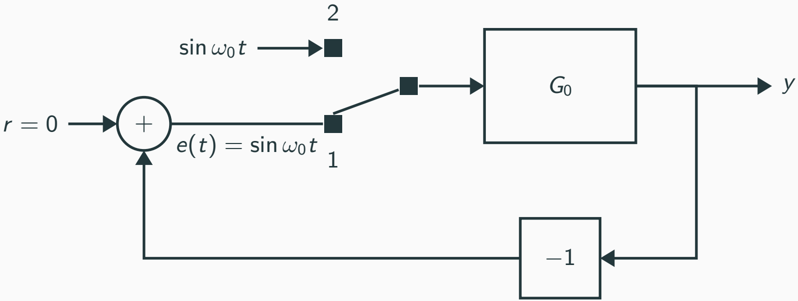


Oscillation will continue in closed loop. We have a marginally stable system.

Seems likely that

- $|G_0(i\omega_0)| < 1 \Rightarrow$ Oscillation damped out (Asymptotic stability)
- $|G_0(i\omega_0)| > 1 \Rightarrow$ Oscillation increases (Instability)

Bode and Nyquist diagrams

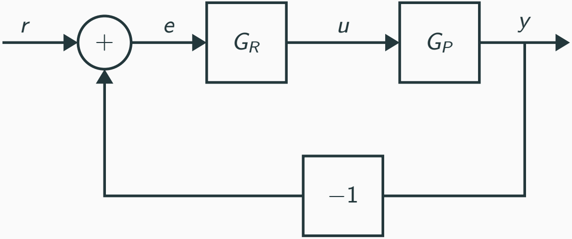
We **most often** plot Bode and Nyquist diagrams for “the open-loop system” G_O (aka *loop gain* L)

$$L = G_O = G_R G_p$$

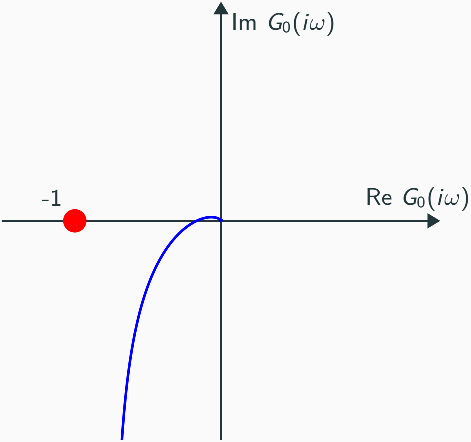
and from this predict how the closed-loop system

$$\frac{G_R G_p}{1 + G_R G_p}$$

will behave.



Nyquist's Criterion

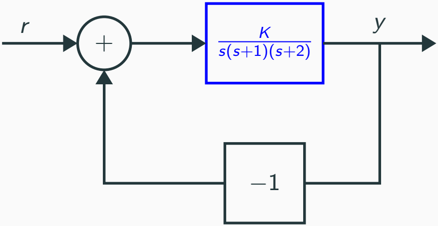


Nyquist's Criterion (simplified version):

Assume $G_0(s)$ is stable.

Then the closed loop system (simple negative feedback) is stable if the point -1 lies to the left of $G(i\omega)$ as ω goes from 0 to ∞.

Example



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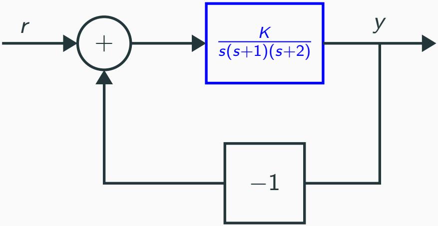
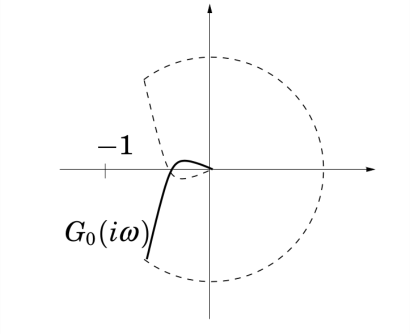


Loop gain (Open system)

$$
\begin{aligned}
G_0(i\omega) &= \frac{K}{i\omega(1+i\omega)(2+i\omega)} \\
&= \frac{-Ki(1-i\omega)(2-i\omega)}{\omega(1+\omega^2)(4+\omega^2)} = \frac{-Ki(2-\omega^2-3i\omega)}{\omega(1+\omega^2)(4+\omega^2)} \\
&= \frac{-3K}{(1+\omega^2)(4+\omega^2)} + i\frac{K(\omega^2-2)}{\omega(1+\omega^2)(4+\omega^2)}
\end{aligned}
$$

$$\lim_{R\to\infty} G_0(Re^{i\phi}) = 0 \qquad\qquad \lim_{r\to 0} G_0(re^{i\phi}) = \frac{K}{2r}e^{-i\phi}$$

Stability for closed-loop system



Crossing with negative real axis:

$$\text{Phase} = \text{-180 deg} \Longrightarrow \mathrm{Im}\left\{G_O(i\omega_0)\right\} = 0 \Longrightarrow \underline{\omega_0 = \sqrt{2}}$$

$$G_0(i\sqrt{2}) = -\frac{3K}{3 \cdot 6} = -\frac{K}{6}$$

Stable if $K < 6$. Two poles in right half-plane if $K > 6$.

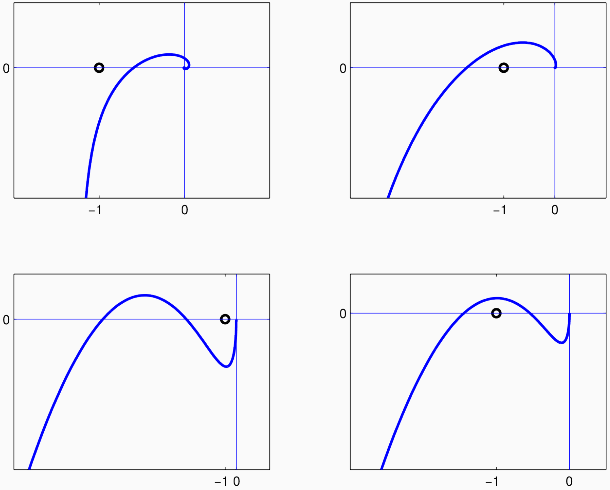
Nyquist's criterion — Some comments

- Gives insight
- Easy to use, only requires frequency response
- Slightly complex to prove
- Version of Nyquist Criterion also works if $G_0(s)$ is unstable.

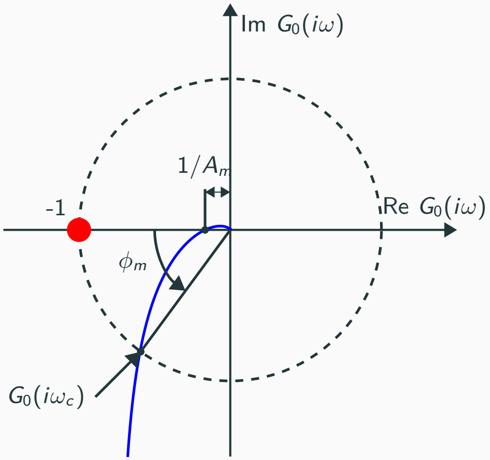
Quiz

Nyquist curves of four (open-loop stable) systems.

Which systems are stable in closed loop (simple negative feedback)?



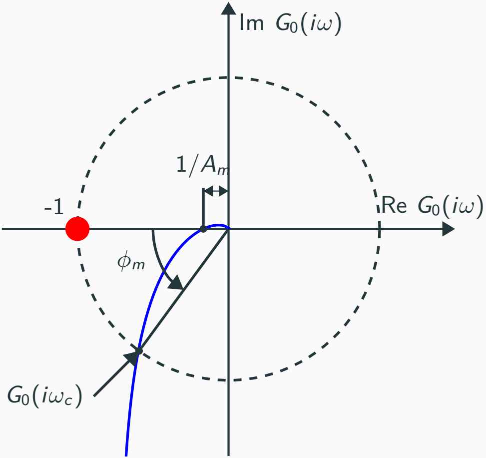
Stability Margin



Amplitude margin: "Gain increase without instability"

Phase margin: "Phase decrease without instability"

Stability Margin



Important with sufficient stability margins for good control performance

Rule of thumb: $A_m > 2$, $\phi_m > 45°$

Delay Margin

Augment open-loop transfer function $G_0(s)$ with a delay L:

$$G_0^{new}(s) = e^{-sL} G_0(s)$$

We have

$$|G_0^{new}(i\omega)| = |G_0(i\omega)|$$

$$\arg G_0^{new}(i\omega) = \arg G_0(i\omega) - \omega L$$

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Same cross-over frequency ω_c as G_0, so new phase margin

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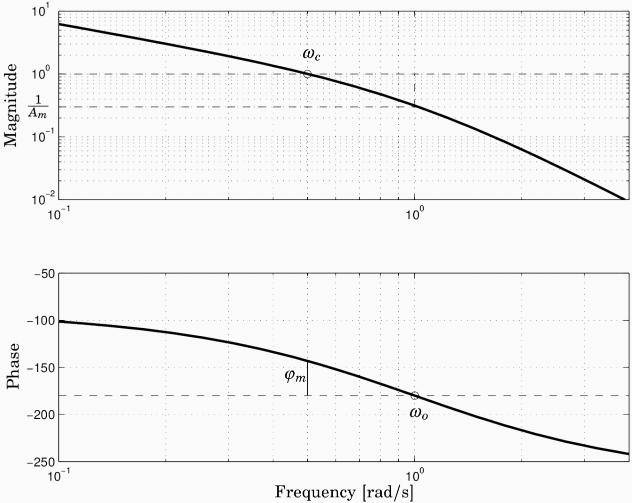
For stability the delay L must be smaller than

$$L_m = \frac{\varphi_m}{\omega_c}$$

Amplitude & Gain Margins in Bode Plots



ω_c is called the cross-over frequency.

The Sensitivity Function

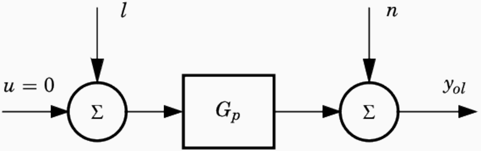
The closed-loop transfer function

$$S(s) = \frac{1}{1 + G_R(s)G_P(s)}$$

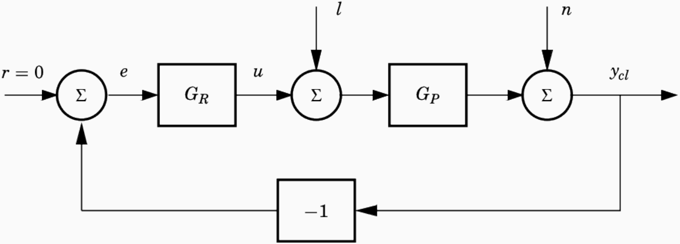
is called the **sensitivity function**.

Gives much information about closed-loop control performance.

Interpretation of Sensitivity Function (1/3)



$$Y_{ol}(s) = \ldots\ L(s) + \ldots\ N(s)$$



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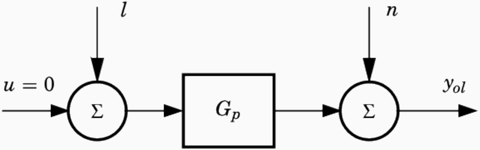
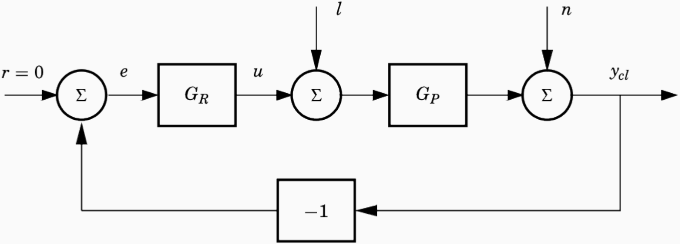


$$Y_{cl}(s) = G_P(s)L(s) + 1 \cdot N(s)$$



$$Y_{cl}(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)}L(s) + \frac{1}{1 + G_R(s)G_P(s)}N(s)$$

Interpretation of Sensitivity Function (1/3)

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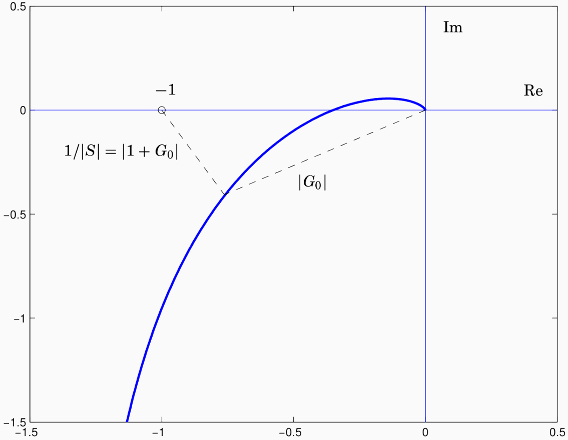
$$Y_{cl}(s) = \frac{G_P(s)}{1 + G_R(s)G_P(s)}L(s) + \frac{1}{1 + G_R(s)G_P(s)}N(s)$$

The sensitivity function quantifies the effect of feedback.

$|S(i\omega)| < 1 \Rightarrow$ disturbances with frequency ω are reduced by controller
$|S(i\omega)| > 1 \Rightarrow$ disturbances with frequency ω are magnified by controller

Typically the controller will always increase disturbances at some frequencies. Preferably not at frequencies with much disturbances.

Interpretation of Sensitivity Function (2/3)



$1/|S(i\omega)|$ is the distance between the Nyquist curve and -1.

$M_s = \sup_\omega |S(i\omega)|$ can be used to quantify the stability margin.

Interpretation of Sensitivity Function (3/3)

The sensitivity function quantifies closed-loop sensitivity to modeling errors. Let G_P be our process model.

$$G_P^0 = G_P(1 + \Delta G)$$

G_P^0 is the actual process dynamics, ΔG is the relative modeling error .

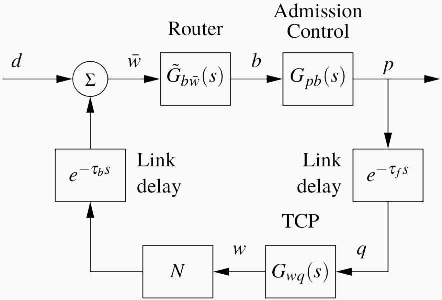
Can show that

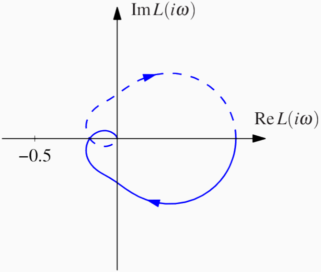
$$Y^0 = \left(1 + S^0 \Delta G\right) Y$$

S^0 is the sensitivity function of the *real* system.

$$\frac{Y^0 - Y}{Y} = S^0 \Delta G$$

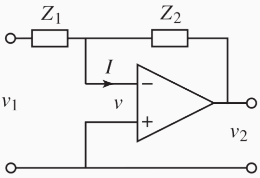
Example: Internet Congestion Control



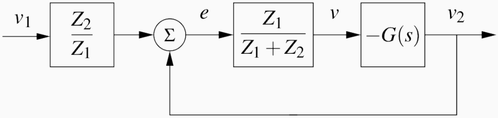


See Example 9.5 in [Åström & Murray] for details.

Example: Operational Amplifier



(a) Amplifier circuit



(b) Block diagram

Transfer function from v_1 to v_2;

$$G_{cl}(i\omega) = -\frac{Z_2}{Z_1}\frac{Z_1 G(i\omega)/(Z_1+Z_2)}{1+Z_1 G(i\omega)/(Z_1+Z_2)}$$

$\approx -Z_2/Z_1$ (If closed loop is stable, and ω within bandwidth)

What about stability? Just look at Nyquist curve of

$$G_o(s) = \frac{Z_1 G(s)}{Z_1+Z_2}$$

Don't need model of the op-amp, just measured transfer function!
(Power of Nyquist's Criterion)

Extra

Cauchy's argument variation principle

How many zeros does a rational function $f(\cdot)$ have in a region C?

$$\frac{1}{2\pi}\Delta_{s\in C}\arg f(s) = P - N$$

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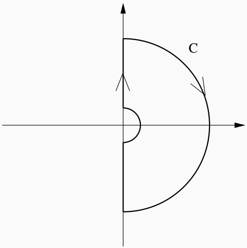
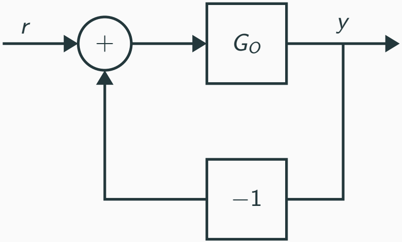
To determine the number of roots in the right half plane we choose the closed curve C in the following way.



Half-circle around the origin avoids singularities on the boundary

Stability for feedback

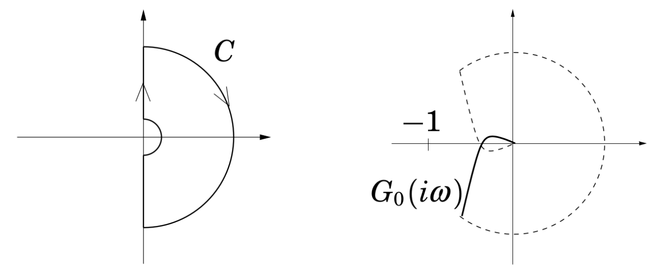


The closed-loop system is asymptotically stable if and only if all zeros to

$$1 + G_0(s)$$

are in the left half-plane.

Cauchy's argument variation principle for feedback



$$N = \#\text{ zeros for } 1 + G_0(s) \text{ inside curve } C$$

$$P = \#\text{ poles for } 1 + G_0(s) \text{ inside curve } C$$

Argument variation principle gives

$$\begin{aligned} P - N &= \#\text{ rev. around origin for } 1 + G_0(s),\ s \in C \\ &= \#\text{ rev. around } -1 + 0i \text{ for } G_0(i\omega),\ \omega \in \mathbf{R} \end{aligned}$$

Nyquist criterion

If $G_0(s)$ is stable ($P = 0$), then the closed-loop system $[1 + G_0(s)]^{-1}$ is stable ($N = 0$) if and only if the Nyquist-curve $G(i\omega)$ does NOT encircle $-1 + 0i$.

The difference between the number of unstable poles in $G_0(s)$ and the number of unstabila poles in $[1 + G_0(s)]^{-1}$ is equal to the number of turns of the Nyquist-curve around $-1 + 0i$.