Linearization, Transfer Function, Block Diagram Representation, Transient Response

Automatic Control, Basic Course, Lecture 2

November 8, 2017

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Content

1. Linearization

2. Transfer Function

3. Block Diagram Representation

4. Transient Response

From Lecture 1

- PID-control
- State-space model of plant

Linearization

Linearization - Why?

Many systems are nonlinear. However, one can approximate them with linear ones. This to get a system that is easier to analyze.

A few examples of nonlinear systems:

- Water tanks (Lab 2)
- Air resistance
- Action potentials in neurons
- Pendulum under the influence of gravity
- ...

Linearization - How?

Given a nonlinear system $\dot{x} = f(x, u),\ y = g(x, u)$

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2. Make a first order Taylor series expansions of f and g around (x_0, u_0):

$$f(x, u) \approx f(x_0, u_0) + \frac{\partial}{\partial x} f(x_0, u_0)(x - x_0) + \frac{\partial}{\partial u} f(x_0, u_0)(u - u_0)$$
$$g(x, u) \approx g(x_0, u_0) + \frac{\partial}{\partial x} g(x_0, u_0)(x - x_0) + \frac{\partial}{\partial u} g(x_0, u_0)(u - u_0)$$

   Notice that $f(x_0, u_0) = 0$ and let $y_0 = g(x_0, u_0)$

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   Notice that $f(x_0, u_0) = 0$ and let $y_0 = g(x_0, u_0)$

3. Introduce $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$

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   Notice that $f(x_0, u_0) = 0$ and let $y_0 = g(x_0, u_0)$

3. Introduce $\Delta x = x - x_0$, $\Delta u = u - u_0$ and $\Delta y = y - y_0$
4. The state-space equations in the new variables are given by:

$$\dot{\Delta x} = \dot{x} - \dot{x}_0 = f(x, u) \approx \frac{\partial}{\partial x} f(x_0, u_0)\Delta x + \frac{\partial}{\partial u} f(x_0, u_0)\Delta u = A\Delta x + B\Delta u$$

$$\Delta y = g(x, u) - y_0 \approx \frac{\partial}{\partial x} g(x_0, u_0)\Delta x + \frac{\partial}{\partial u} g(x_0, u_0)\Delta u = C\Delta x + D\Delta u$$

Example - Linearization

Example

The dynamics of a specific system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} \\ y &= x_1^2 + u^2 \end{aligned}$$

a) Find all stationary points

b) Linearize the system around the stationary point corresponding to $u_0 = 3$

The dynamics of a specific system is described by

$$
\begin{aligned}
\dot{x}_1 &= x_2 && = f_1(x_1,\ x_2,\ u) \\
\dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && = f_2(x_1,\ x_2,\ u) \\
y &= x_1^2 + u^2 && = g(x_1,\ x_2,\ u)
\end{aligned}
$$

(a) Find stationary point for $u_0 = 3$: $(\dot{x}_1 = \dot{x}_2 = 0)$

$$
\begin{aligned}
\mathbf{0} &= x_2 \\
\mathbf{0} &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{\mathbf{3} + 1} \\
y &= x_1^2 + \mathbf{3^2}
\end{aligned}
$$

$$
\Longrightarrow (\mathbf{x_{10}},\ \mathbf{x_{20}}, \mathbf{u_0}) = (-2,\ 0,\ 3)
$$

$$
y_0 = g(x_{10}, x_{20}, u_0) = 13
$$

The dynamics of a specific system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 && = f_1(x_1,\, x_2,\, u) \\ \dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && = f_2(x_1,\, x_2,\, u) \\ y &= x_1^2 + u^2 && = g(x_1,\, x_2,\, u) \end{aligned}$$

$$\Longrightarrow (\mathbf{x_{10}},\, \mathbf{x_{20}}, \mathbf{u_0}) = (-2,\, 0,\, 3)$$

$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

(b) Linearize around stationary point $(-2,\, 0,\, 3)$

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 0, && \frac{\partial f_1}{\partial x_2} = 1, && \frac{\partial f_1}{\partial u} = 0, \\ &\frac{\partial f_2}{\partial x_1} = +3\frac{x_2^4}{x_1^3} + 1, && \frac{\partial f_2}{\partial x_2} = -4\frac{x_2^3}{x_1^2}, && \frac{\partial f_2}{\partial u} = \frac{1}{2\sqrt{u+1}}, \\ &\frac{\partial g}{\partial x_1} = 2x_1, && \frac{\partial g}{\partial x_2} = 0, && \frac{\partial g}{\partial u} = 2u, \end{aligned}$$

The dynamics of a specific system is described by

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y &= x_1^2 + u^2 && = g(x_1,\, x_2,\, u)
\end{aligned}$$

$$\Longrightarrow (\mathbf{x_{10}},\, \mathbf{x_{20}}, \mathbf{u_0}) = (-2,\, 0,\, 3)$$
$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

(b) Linearize around stationary point $(-2,\, 0,\, 3)$

$$\begin{aligned}
\frac{\partial f_1}{\partial x_1}_{|\{x_0,\, u_0\}} &= 0, & \frac{\partial f_1}{\partial x_2}_{|\{x_0,\, u_0\}} &= 1, & \frac{\partial f_1}{\partial u}_{|\{x_0,\, u_0\}} &= 0, \\
\frac{\partial f_2}{\partial x_1}_{|\{x_0,\, u_0\}} &= 1, & \frac{\partial f_2}{\partial x_2}_{|\{x_0,\, u_0\}} &= 0, & \frac{\partial f_2}{\partial u}_{|\{x_0,\, u_0\}} &= \frac{1}{4}, \\
\frac{\partial g}{\partial x_1}_{|\{x_0,\, u_0\}} &= -4, & \frac{\partial g}{\partial x_2}_{|\{x_0,\, u_0\}} &= 0, & \frac{\partial g}{\partial u}_{|\{x_0,\, u_0\}} &= 6,
\end{aligned}$$

The dynamics of a specific system is described by

$$\begin{aligned}\dot{x}_1 &= x_2 && = f_1(x_1,\, x_2,\, u)\\ \dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} && = f_2(x_1,\, x_2,\, u)\\ y &= x_1^2 + u^2 && = g(x_1,\, x_2,\, u)\end{aligned}$$

$$\Longrightarrow (\mathbf{x_{10}},\, \mathbf{x_{20}}, \mathbf{u_0}) = (-2,\, 0,\, 3)$$
$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

(b) Linearize around stationary point $(-2,\, 0,\, 3)$

$$\frac{f(x,u)}{\partial x}_{|\{x_0,\, u_0\}} = A \quad = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \frac{f(x,u)}{\partial u}_{|\{x_0,\, u_0\}} = B \quad = \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix}$$

$$\frac{g(x,u)}{\partial x}_{|\{x_0,\, u_0\}} = C \quad = \begin{bmatrix} -4 & 0 \end{bmatrix} \qquad \frac{g(x,u)}{\partial u}_{|\{x_0,\, u_0\}} = D \quad = \begin{bmatrix} 6 \end{bmatrix}$$

The dynamics of a specific system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 & &= f_1(x_1,\, x_2,\, u) \\ \dot{x}_2 &= -\frac{x_2^4}{x_1^2} + x_1 + \sqrt{u+1} & &= f_2(x_1,\, x_2,\, u) \\ y &= x_1^2 + u^2 & &= g(x_1,\, x_2,\, u) \end{aligned}$$

$$\Longrightarrow (\mathbf{x_{10}},\, \mathbf{x_{20}}, \mathbf{u_0}) = (-2,\, 0,\, 3)$$
$$y_0 = g(x_{10}, x_{20}, u_0) = 13$$

Introduce

$$\Delta x_1 = x_1 - x_{10}, \qquad \Delta x_2 = x_2 - x_{20}$$
$$\Delta u = u - u_0 \qquad \Delta y = y - y_0$$

The state-space equations in the new variables are given by:

$$\begin{bmatrix} \frac{\Delta x_1}{dt} \\ \frac{\Delta x_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} u$$
$$\Delta y = \begin{bmatrix} -4 & 0 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \end{bmatrix} + \begin{bmatrix} 6 \end{bmatrix} u$$

Transfer Function

Laplace Transformation

Let $f(t)$ be a function of time t, the Laplace transformation $\mathcal{L}(f(t))(s)$ is defined as

$$\mathcal{L}(f(t))(s) = F(s) = \int_0^\infty e^{-st} f(t) \mathrm{d}t$$

Example:

$$\mathcal{L}\left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)(s) = sF(s) - f(0)$$

Initial values helps to calculate what happens in transient phase!

Assuming that $f(0) = f'(0) = \cdots = f^{n-1}(0) = 0$ (common assumption during this course, but not always!!) it has the property that

$$\mathcal{L}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right)(s) = s^n F(s)$$

$$\mathcal{L}\left(\int_0^t f(\tau)\frac{de}{d\tau}\right)(s) = \frac{1}{s}F(s) \qquad (integrator)$$

See Collection of Formulae for a table of Laplace transformations.

Example - Transfer Function

Example

A system's dynamics is described by the differential equation

$$\ddot{y} + a_1\dot{y} + a_2y = b_1\dot{u} + b_2u.$$

After Laplace transformation we get

$$(s^2 + a_1s + a_2)Y(s) = (b_1s + b_2)U(s)$$

which can be written as

$$Y(s) = \overbrace{\frac{b_1s + b_2}{s^2 + a_1s + a_2}}^{G(s)}U(s) = G(s)U(s)$$

$G(s)$ is called the transfer function of the system.

Transfer Function

Relation between control signal $U(s)$ and output $Y(s)$:

$$Y(s) = G(s)U(s)$$

$G(s)$ often fraction of polynomal, i.e.,

$$G(s) = \frac{Q(s)}{P(s)}$$

Zeros of $Q(s)$ are called zeros of the system, zeros of $P(s)$ are called poles of the system.

The poles play a very important role for the system's behavior.

From State Space to Transfer Function

For a system on state space form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

the transfer function is given by

$$G(s) = C(sI - A)^{-1}B + D$$

Observe: the denominator of $G(s)$ is given by $P(s) = \det(sI - A)$, so eigenvalues of A are poles of the system.

From Transfer Function to State Space

Can be done in several ways, see Collection of Formulae.

Example

A system's transfer function is

$$G(s) = \frac{2s + 1}{s^3 + 4s - 8}$$

Write the system on a state space form of your choice.

Three Ways to Describe a Dynamical System

$$\begin{tikzcd}
 & \begin{array}{c}\text{Differential equation}\\ \ddot{y}+a_1\dot{y}+a_2y=b_1\dot{u}+b_2u\end{array} \arrow[dl, bend right, "\begin{array}{l}x_1=y\\ x_2=\dot{y}\\ \dots\end{array}"'] \arrow[dr, leftrightarrow, bend left, "\mathcal{L}\left(\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}\right)(s)=s^nF(s)"] & \\
\begin{array}{c}\text{State space}\\ \dot{x}=Ax+Bu\\ y=Cx+Du\end{array} \arrow[rr, bend right, "G(s)=C(sI-A)^{-1}B+D"'] & & \begin{array}{c}\text{Transfer function}\\ Y(s)=G(s)U(s)=\frac{Q(s)}{P(s)}U(s)\end{array} \arrow[ll, bend right, "\text{Collection of Formulae}"']
\end{tikzcd}$$

Block Diagram Representation

Block Diagram - Transfer Function

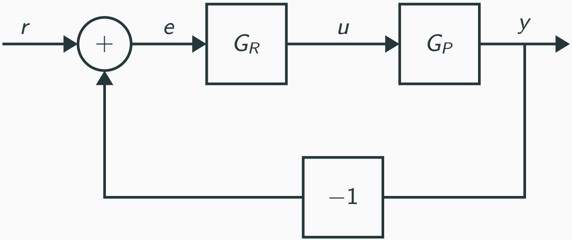
When the blocks in a block diagram are replaced by transfer functions, it is possible to describe the relations between signals in an easy way.

$$Y(s) = G_P(s)U(s)$$

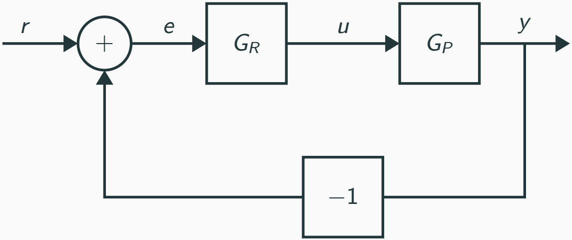
Block Diagram - Components

Most block diagrams consist of three components:

- Blocks - Transfer functions
- Arrows - Signals
- Summations



Determine Transfer Function From Block Diagram



$$Y = G_P U, \quad U = G_R E, \quad E = R - Y$$

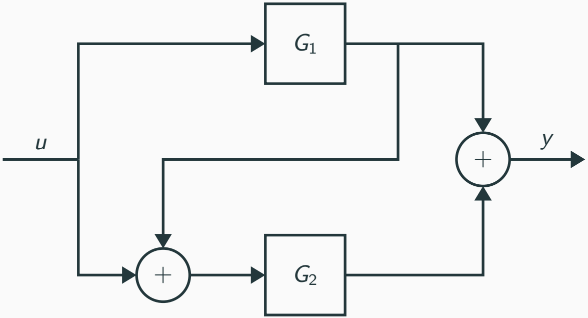
From the equations above the transfer function between r and y is

$$Y = \frac{G_P G_R}{1 + G_P G_R} R$$

Example - Transfer Functions

Example

Two systems, G_1 and G_2, are interconnected as in the figure below



Compute the transfer function from u to y, G_{yu}.

Transient Response

Solution to State Space Equation

Given a system on state space form

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The solution, $y(t)$, is then given by

$$y(t) = Ce^{At}x(0) + C\int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau + Du(t)$$

Initial state, uninteresting except when the controller is initialized

Weighted integral of the control signal, interesting part

Direct term, often neglectable in practical systems

Impulse Response

Shows how the system responds when the input is a short pulse, i.e., a Dirac function

$$u(t) = \delta(t)$$

The Laplace transformation is

$$U(s) = \int_0^\infty e^{-st}\delta(t)\mathrm{d}t = 1$$

Hence

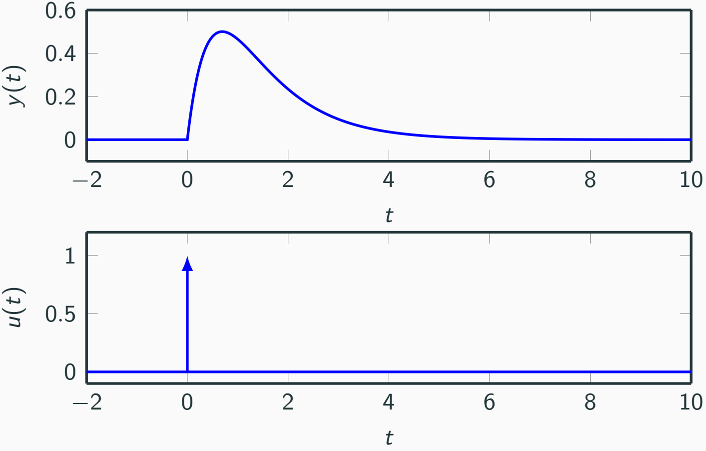
$$Y(s) = G(s)U(s) = G(s)$$

Not so common in technological applications, can we think of other applications?

Example - Impulse Response

Let the transfer function of the system be:

$$G(s) = \frac{2}{s^2 + 3s + 2}$$



Step Response

Shows how the system responds when the input is a step, i.e.,

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The Laplace transformation is

$$U(s) = \int_0^\infty e^{-st} u(t) \mathrm{d}t = \int_0^\infty e^{-st} \mathrm{d}t = -\frac{1}{s} \left[e^{-st} \right]_0^\infty = \frac{1}{s}$$

Very common in technological applications

Example - Step Response

Let the transfer function of the system be:

$$G(s) = \frac{2}{s^2 + 3s + 2}$$