LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Automatic Control, Basic Course (FRT010)

2105-12-17

Points and grades

All answers must include a clear motivation. The total number of points is 25. The maximum number of points is specified for each subproblem.

Grade 3: 12 points
4: 17 points
5: 22 points

Accepted aid

Mathematical collections of formulae (e.g. TEFYMA), 'Collections of formulae in automatic control', and calculators that are not programmed in advance.

Results

The graded exam will be displayed on December 19, 08-10 am in building 4, room 303. Thereafter, exams will be archived at the Automatic Control department in Lund.

Solutions for the exam Automatic Control, Basic Course 2015-12-17

1. A certain system is described by the following differential equation:

$$\ddot{y}(t) + 3\dot{y}(t) + 12y(t) + 2\ddot{u}(t) - 5\dot{u}(t) = -2u(t).$$

a. Write the transfer function from the input u to the output y. (1 p)

b. If u is a unit step, what is the stationary value of y? (1 p)

c. What is the order of the system? (0.5 p)

d. Is the system linear or nonlinear? Motivate your answer. (0.5 p)

Solution

a. Laplace transformation of the differential equation gives:

$$s^2Y(s) + 3sY(s) + 12Y(s) + 2s^2U(s) - 5sU(s) = -2U(s),$$

$$Y(s) = \frac{-2s^2 + 5s - 2}{s^2 + 3s + 12}U(s).$$

b. The characteristic polynomial is $s^2 + 3s + 12$, and both its poles have real part -1.5. (For a second order polynomial it is sufficient to check that all coefficients are positive, to ensure roots with negative real part.) The system is consequently asymptotically stable, and the final value theorem can be used to compute the stationary value of y.

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} s \cdot \frac{-2s^2 + 5s - 2}{s^2 + 3s + 12} \cdot \frac{1}{s} = -\frac{1}{6}.$$

c. The system has two poles, of which none can be cancelled by the system zeros. It is therefore of second order.

d. The differential equation is a linear combination of $y(t)$ and $u(t)$ and time derivatives of these. Hence, the system is linear.

2. Consider the nonlinear system

$$\dot{x}_1 = -x_1 + x_2^2,$$
$$\dot{x}_2 = x_1x_2 - u^3.$$

a. Find all stationary points of the system corresponding to $u^0 = 5$. (1 p)

b. Linearize the system around (one of) the stationary points found in the previous step. If you did not solve the previous problem, use the point $({x_1}^0, {x_2}^0, u^0) = (3, 9, 3)$. (2 p)

c. Find the poles of the linearized system and comment upon its stability properties. (2 p)

Solution Introduce $f_1(x,u)$ and $f_2(x,u)$ such that

$$\dot{x}_1 = -x_1 + x_2^2 = f_1(x,u),$$
$$\dot{x}_2 = x_1 x_2 - u^3 = f_2(x,u).$$

a. The stationary points can be found by solving

$$f_1(x^0,u^0) = 0, \quad f_2(x^0,u^0) = 0.$$

Inserting $u^0 = 5$, and starting with the first equation, we get

$$x_1^0 = (x_2^0)^2.$$

Introducing that in the second equation with gives us

$$(x_2^0)^3 = (u^0)^3.$$

from which we see that $x_2^0 = u^0 = 5$. Evaluation of the first equation with $x_2^0 = 5$ gives us $x_1^0 = 25$. There is only one stationary point: $(x_1^0, x_2^0, u^0) = (25, 5, 5)$.

b. The system should be linearized around $(x_1^0, x_2^0, u^0) = (25, 5, 5)$. The partial derivatives needed are

$$\frac{\partial f_1}{\partial x_1} = -1, \qquad \frac{\partial f_1}{\partial x_2} = 2x_2,$$

$$\frac{\partial f_2}{\partial x_1} = x_2, \qquad \frac{\partial f_2}{\partial x_2} = x_1,$$

and

$$\frac{\partial f_1}{\partial u} = 0,$$
$$\frac{\partial f_2}{\partial u} = -3u^2.$$

Introduce the new variables

$$\Delta x = x - x^0,$$
$$\Delta u = u - u^0.$$

Evaluation of the partial derivatives at the stationary point gives the following state space representation of the linearized system:

$$\Delta\dot{x} = A\Delta x + B\Delta u,$$

where

$$A = \begin{bmatrix} -1 & 10 \\ 5 & 25 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -75 \end{bmatrix}.$$

Along the same lines, the matrices corresponding to the stationary point $(x_1^0, x_2^0, u^0) = (3, 9, 3)$ are

$$A = \begin{bmatrix} -1 & -18 \\ 9 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ -27 \end{bmatrix}.$$

c. The poles are given by the eigenvalues of the A-matrix.

$$\det(sI - A) = \begin{vmatrix} s+1 & -10 \\ -5 & s-25 \end{vmatrix} = (s+1)(s-25) - 50 = s^2 - 24s - 75 = 0.$$

By solving this equation we obtain the poles:

$$s = 12 \pm \sqrt{219} = 12 \pm 14.80.$$

One of the poles has positive real part (26.80), so the system is unstable.

3. You are given a physical process with transfer function

$$G_p(s) = \frac{2}{s+3}.$$

Design a PI-controller so that all poles of the closed-loop system are -3. (2 p)

Solution

The open-loop transfer function is

$$G_o = P(s)C(s) = \frac{2K(T_i s + 1)}{T_i s(s+3)}.$$

Denoting the numerator and denominator of G_o by V and W, respectively, the closed-loop transfer function can be written

$$G_c = \frac{G_o}{1+G_o} = \frac{V/W}{1+V/W} = \frac{V}{V+W} = \frac{2K(T_i s+1)}{2K(T_i s+1) + T_i s(s+3)}.$$

The characteristic polynomical (normalized with respect to the coefficient of s^2) is $s^2 + (2K+3)s + 2K/T_i$. Matching this with the specification $(s+3)^2 = s^2 + 6s + 9$ yields

$$\begin{cases} 2K + 3 = 6, \\ \dfrac{2K}{T_i} = 9, \end{cases}$$

with solution $K = 3/2$, and $T_i = 1/3$.

(One can note that the true order of the closed-loop system is one, due to a pole-zero cancellation.)

4. You are given a system

$$\begin{cases} \dot{x} &= \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\ y &= \begin{bmatrix} 1 & 1 \end{bmatrix} x. \end{cases}$$

a. Is the system controllable? (1 p)

b. Design a state feedback controller

$$u = -Lx + l_r r,$$

such that the resulting closed-loop system has its poles in $s = -4$ and $s = -5$. Determine l_r so that y = r in stationarity. (2 p)

c. Is the system observable? (1 p)

d. Suppose that measures of the states are not available. Design a Kalman filter to estimate the value of the states. Place the poles of the Kalman filter in -10. (2 p)

Solution

a. The controllability matrix is given by

$$W_c = [B \quad AB] = \begin{bmatrix} 0 & 1 \\ 1 & -4, \end{bmatrix}$$

and its determinant is $-1 \neq 0$. This means that W_c is full rank, which is equivalent to the system being controllable.

b. The characteristic polynomial of the closed-loop system is the determinant of $sI - (A - BL)$. Therefore we have to find the determinant of

$$\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ l_1 & l_2 \end{bmatrix} \right) = \begin{bmatrix} s-2 & -1 \\ l_1 & s+4+l_2 \end{bmatrix}.$$

The determinant of the above matrix is $s^2 + (l_2+2)s + (l_1 - 2l_2 - 8)$. Matching coefficients of s with the desired characteristic polynomial $(s+4)(s+5) = s^2 + 9s + 20$ yields

$$\begin{cases} l_2 + 2 = 9, \\ l_1 - 2l_2 - 8 = 20, \end{cases}$$

with solution $L = [l_1 \quad l_2] = [42 \quad 7]$.

The transfer function of the closed-loop system is $G(s) = C(sI - (A - BL))^{-1}Bl_r$. Since the system is asymptotically stable by design, we can use the final value theorem to compute the static gain

$$G(0) = C(BL - A)^{-1}Bl_r = [1 \quad 1] \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} [42 \quad 7] - \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} l_r.$$

Performing the numeric computations, we obtain $G(0) = -l_r/20$. That is, $l_r = -20$ gives the static gain $G(0) = 1$.

c. Since

$$W_O = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix},$$

and its determinant is equal to $-5 \neq 0$, the system is observable.

d. The characteristic equation for the Kalman filter is given by $\det(sI - A + KC)$, where $K = [k_1 \quad k_2]^T$ is our design variable. Computing the determinant gives (cf. previous problem for choosing state feedback vector L):

$$\begin{vmatrix} s + k_1 - 2 & k_1 - 1 \\ k_2 & s + k_2 + 4 \end{vmatrix} = s^2 + (k_1 + k_2 + 2)s + (4k_1 - k_2 - 8).$$

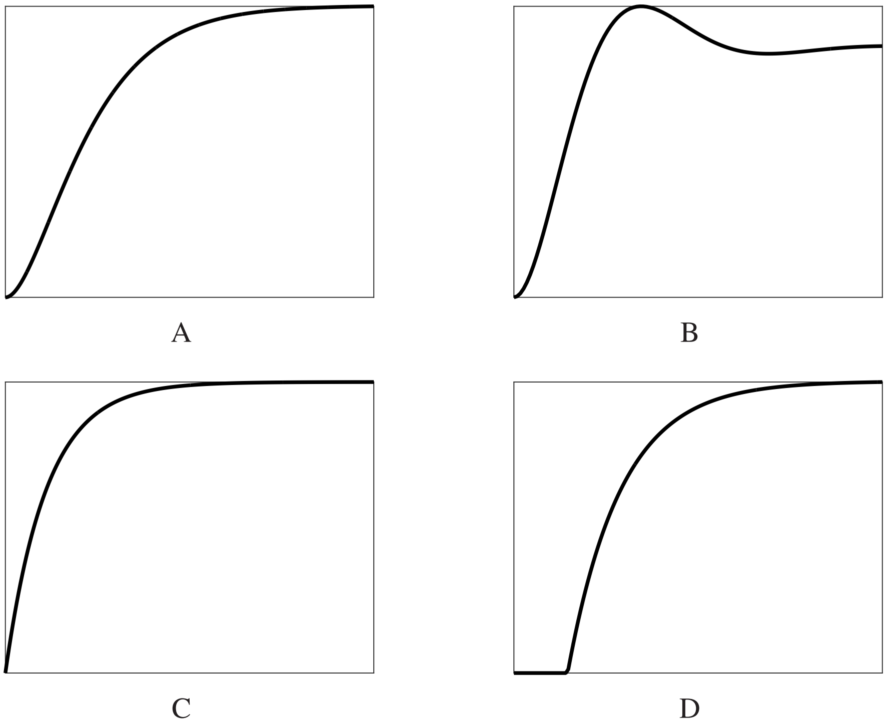
Equating coefficient of s with those of the desired dynamics $(s+10)^2 = s^2 + 20s + 100$ yields

$$\begin{cases} k_1 + k_2 + 2 = 20, \\ 4k_1 - k_2 - 8 = 100. \end{cases}$$

The system has solution $K = [k_1 \quad k_2]^T = \frac{1}{5}[126 \quad -36]^T = [25.2 \quad -7.2]^T$.

5. Pair the following transfer functions and step responses. Do not forget to motivate your answers. (2 p)

$$P_1(s) = \frac{1}{s+1}, \quad P_2(s) = \frac{1}{s+1}e^{-s}, \quad P_3(s) = \frac{1}{(s+1)^2}, \quad P_4(s) = \frac{1}{s^2+s+1}.$$



Solution

- **2–D**. P_2 is the only transfer function with a pure time delay, as seen in step response D.
- **1–C**. Both P_3 and P_4 are asymptotically stable second order systems. The initial value theorem applied to the derivative of their step response results in 0. Since the initial derivative of C is non-zero, it must consequently correspond to P_1.
- **3–A**. P_3 has real poles, resulting in a critically damped step response.
- **4–B**. The characteristic polynomial of P_4 corresponds to angular speed $\omega = 1$ and relative damping $\zeta = 1/2$. It therefore results in the only oscillatory step response, B.

6. Are there situations where feed forward is preferential to feedback control? If so, give an example. (1 p)

Solution

Feed forward is efficient in eliminating known or directly measurable disturbances, before they appear in the process output (which is a necessity for feedback to attenuate them).

One example is having access to outdoor temperature measurement when controlling indoor temperature, and establish a feed forward link, which for instances increases radiator output at night, before a dip in indoor temperature arises.

7. You are faced with designing a P controller $C(s) = K$ for a process with dynamics

$$P(s) = \frac{2}{s(s+1)}.$$

You would like to maximize K, in order to minimize the stationary error due to load disturbances. However, due to model uncertainty, you need to maintain a 30° phase margin. What value of K do these design criteria result in? (3 p)

Solution

The P controller leaves the open-loop phase unaffected, while changing its gain by a factor K. We therefore need to compute the process gain at the phase shift corresponding to the desired phase margin, in order to determine K.

The phase of the process is given by

$$\arg\left(P(i\omega)\right) = \arg\left(\frac{2}{i\omega(i\omega+1)}\right) = \arg(2) - \arg(i\omega+1) - \arg(i\omega) = -\arctan(\omega) - 90^\circ.$$

Equating this with the desired phase shift

$$-180^\circ + \varphi_m = -180^\circ + 30^\circ = -150^\circ,$$

gives the corresponding angular frequency

$$-150^\circ = -\arctan(\omega) - 90^\circ \Rightarrow \omega = \tan(60^\circ) = \sqrt{3}.$$

The gain of the process at this frequency is

$$|P(i\omega)| = \frac{2}{|\omega|\sqrt{\omega^2+1}} = \frac{2}{\sqrt{3}\sqrt{3+1}} = \frac{1}{\sqrt{3}},$$

and the maximal admissible controller gain is

$$K = |P(i\omega)|^{-1} = \sqrt{3} \approx 1.73.$$

8. You are introduced to a control system with open-loop transfer function

$$G_o(s) = \frac{10}{s(s+1)}.$$

Performance is satisfactory in terms of robustness. However, your employer would like to make the control loop faster. Propose a solution that doubles the cross-over frequency, while maintaining the current phase margin. (3 p)

Solution

There are several ways to solve this problem. Here we will use the lead compensation link

$$G_K(s) = K_k N \frac{s+b}{s+bN},$$

introduced in the course. First we compute the current cross-over frequency ω_c^0, by solving

$$|P(i\omega_c^0)| = \frac{10}{\omega_c^0\sqrt{(\omega_c^0)^2+1}} = 1.$$

The above corresponds to a quadratic equation with positive solution

$$\omega_c^0 = \sqrt{\frac{\sqrt{401}-1}{2}} \approx 3.08.$$

The sepcification is to double the cross-over frequency, $\omega_c = 2\omega_c^0 \approx 6.17$.

The phase shift at the current cross-over frequency is

$$\arg\left(G_o(i\omega_c^0)\right) = \arg(10) - \arg(i\omega_c^0+1) - \arg(i\omega_c^0) = -\arctan(\omega_c^0) - 90°.$$

In order to maintain the current phase margin, we need a phase advance

$$\begin{aligned}\Delta\varphi_m &= \left(180° + \arg\left(G_o(i\omega_c^0)\right)\right) - \left(180° + \arg\left(G_o(2i\omega_c^0)\right)\right)\\ &= \arctan(2\omega_c^0) - \arctan(\omega_c^0) \approx 8.76°.\end{aligned}$$

From the graph in the collection of formulae, we see that this rougly corresponds to $N = 1.4$. We choose b to solve

$$\omega_c = b\sqrt{N} \Rightarrow b = \frac{\omega_c}{\sqrt{N}} \approx 5.2,$$

in order to achieve maxium phase advance at ω_c. Finally, we ensure that ω_c becomes the cross-over frequency, by solving

$$|G_K(i\omega_c)||G_o(i\omega_C)| = 1.$$

Here we use that $|G_K(i\omega_c)| = K_K\sqrt{N}$:

$$K_K = \frac{1}{\sqrt{N}|G_o(i\omega_c)|} = \frac{\omega_c\sqrt{\omega_c^2+1}}{10\sqrt{N}} \approx 3.3.$$

(With numerical values given above, the new phase margin is 18.7°, compared with the original one of 18.0°. The difference is a result of numerical rounding combined with a slightly inaccurate reading of the collection of formulae graph when obtaining N.)