1 a. The transfer function from u to y is given by $G(s) = C(sI - A)^{-1}B$.

$$G(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+7 & -2 \\ 0 & s-4 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{2}{(s+7)(s-4)}$$

b. The poles of the system are the eigenvalues of the system matrix A, which are also the zeros of the transfer function denominator polynomial. Solving $(s+7)(s-4) = 0$ shows that the system has one pole in -7 and one pole in 4. Since one pole has positive real part, the system is unstable.

2 a. The stationary points are given by setting $\dot{x}_1 = \dot{x}_2 = 0$, which gives

$$\begin{aligned} 0 &= -2x_1^0 + 1^2 \\ 0 &= -x_1^0 + \sqrt{x_2^0} \end{aligned}$$

Which gives the stationary point as $(u^0, x_1^0, x_2^0) = (1, 0.5, 0.25)$.

b. Calculating the partial derivatives of the two equations (denoted f_1 and f_2) gives

$$\frac{\partial f_1}{\partial x_1} = -2, \qquad \frac{\partial f_1}{\partial x_2} = 0, \qquad \frac{\partial f_1}{\partial u} = 2u,$$
$$\frac{\partial f_2}{\partial x_1} = -1, \qquad \frac{\partial f_2}{\partial x_2} = \frac{1}{2\sqrt{x_2}}, \qquad \frac{\partial f_2}{\partial u} = 0$$

Inserting the stationary point $(u^0, x_1^0, x_2^0) = (1, 0.5, 0.25)$ gives

$$\frac{\partial f_1}{\partial x_1}(1, 0.5, 0.25) = -2, \quad \frac{\partial f_1}{\partial x_2}(1, 0.5, 0.25) = 0, \quad \frac{\partial f_1}{\partial u}(1, 0.5, 0.25) = 2,$$
$$\frac{\partial f_2}{\partial x_1}(1, 0.5, 0.25) = -1, \quad \frac{\partial f_2}{\partial x_2}(1, 0.5, 0.25) = 1, \quad \frac{\partial f_2}{\partial u}(1, 0.5, 0.25) = 0$$

Introduce new variables

$$\begin{aligned} \Delta x &= x - x^0 \\ \Delta u &= u - u^0 \end{aligned} \tag{1}$$

The linearized system is then given by

$$\Delta \dot{x} = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix} \Delta x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Delta u$$

3 a. A-3, B-1, C-2. It is clear that response A lacks integral action since it gives a stationary error when subject to a constant load disturbance, and the system should therefore only contain 4 poles (no integral pole). Both responses B and C contains integral action (no stationary error), and the remaining sets of poles contain 5 poles. The complex poles are the same for all three systems, but the second set of poles also has a slow pole located in -5, it corresponds to the comparably slow response C. Similarly, the third set of poles has a fast pole in -25, which gives the fast response B.

b. By calculating the absolute value of the poles, ω_0 is given. Looking at the angle φ of the poles from the negative real axis, $\zeta = \cos\varphi$ can be calculated.

$$\begin{aligned}
\omega_0^1 &= |-17.1 + 8.3i| = \sqrt{17.1^2 + 8.3^2} \approx 19 \\
\omega_0^2 &= |-14.4 + 7.0i| = \sqrt{14.4^2 + 7^2} \approx 16 \\
\zeta_1 &= \cos\left(\arctan\left(\frac{8.3}{17.1}\right)\right) \approx 0.9 \\
\zeta_2 &= \cos\left(\arctan\left(\frac{7.0}{14.4}\right)\right) \approx 0.9
\end{aligned}$$

c. Increasing ω_0 will in turn increase the gain of the controller, which will reduce stationary errors.

d. Choosing a high value of ω_0 will result in a large control signal, and for this process the control signal is limited to 10 V.

4 a. The low frequency asymptote is given by $G_{LF}(s) = K = 10$. There are three corner frequencies: $\omega_1 = 0.1$ rad/s, $\omega_2 = 1$ rad/s and $\omega_3 = 100$ rad/s. The gain curve breaks downwards once at ω_1, upwards once at ω_2, and downwards once at ω_3, which indicates that there should be poles corresponding to ω_1 and ω_3 and a zero corresponding to ω_2. The phase decreases at ω_1 and ω_3 and increases at ω_2, which confirms this statement. The transfer function thus becomes:

$$G(s) = \frac{10\,(1+s)}{\left(1+\frac{s}{0.1}\right)\left(1+\frac{s}{100}\right)} = \frac{100\,(s+1)}{(s+0.1)\,(s+100)}$$

b. The process output is given by

$$y(t) = 0.1|G(i\omega)|\sin\left(\omega t + \arg(G(i\omega))\right)$$

Studying the Bode diagram at $\omega = 10$ rad/s gives

$$|G(10i)| \approx 1, \quad \arg(G(10i)) \approx -11^\circ.$$

The gain and phase may also be obtained from the transfer function that was determined in subproblem **a**. This gives:

$$\begin{aligned}
|G(10i)| &= \frac{100\sqrt{10^2+1}}{\sqrt{10^2+0.1^2}\sqrt{10^2+100^2}} = 1 \\
\arg(G(10i)) &= \arctan 10 - \arctan 100 - \arctan 0.1 \approx -0.19 \text{ rad} \approx -11^\circ
\end{aligned}$$

The process output becomes

$$y(t) = 0.1\sin(10t - 11^\circ)$$

c. The phase margin is defined at the frequency ω_c where $|G_0(i\omega_c)| = 1$. If we denote the process shown in the Bode plot $G_P(s)$, we get $|2 \cdot G_P(i\omega_c)| = 1$, i.e. $|G_P(i\omega_c)| = 0.5$. By studying the bode magnitude plot we see that $\omega_c \approx 170$ rad/s. The phase of the process at this frequency is approximately $-60°$. Since a P controller only changes the gain of the loop transfer function G_0 and leaves the phase unchanged, the phase margin is given by

$$\phi_m = \pi + \arg G_0(i\omega_c) = \pi + \arg G_P(i\omega_c) = 180° - 60° = 120°$$

when the system is controlled with a P controller with a gain of 2.

Another option is to use the transfer function that was determined in subproblem **a.** to compute the phase margin:

$$\begin{aligned} &\left| 2 \cdot \frac{100\,(i\omega_c + 1)}{(i\omega_c + 0.1)\,(i\omega_c + 100)} \right| = 1 \Rightarrow \omega_c \approx 173 \\ &\arg G_P(i\omega_c) \approx -60° \Rightarrow \phi_m = 120° \end{aligned}$$

5 a. The closed-loop poles are given by the characteristic polynomial

$$\det(sI - A + BL) = \begin{vmatrix} s+1+3 & -1+2 \\ 0 & s+4 \end{vmatrix} = (s+4)(s+4) = 0$$

Thus, the closed-loop system has two poles in -4.

b. The characteristic equation for the Kalman filter is given by

$$\begin{aligned} \det(sI - A + KC) = \begin{vmatrix} s+1+k_1 & -1 \\ k_2 & s+4 \end{vmatrix} &= (s+1+k_1)(s+4) = \\ &= s^2 + (5+k_1)s + 4 + 4k_1 + k_2 \end{aligned}$$

The desired characteristic polynomial is given by $(s+8)(s+8) = s^2+16s+64$. This gives

$$\begin{aligned} 5 + k_1 = 16 &\Rightarrow k_1 = 11 \\ 4 + 4k_1 + k_2 = 64 &\Rightarrow k_2 = 16 \end{aligned}$$

The Kalman filter gain is thus given by $K = \begin{pmatrix} 11 \\ 16 \end{pmatrix}$.

c. If we should be able to place the poles of the closed-loop system arbitrarily using state feedback, the system has to be controllable. The controllability matrix is given by

$$W_c = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Since the controllability matrix does not have two linearly independent columns ($\det W_c = 0$), the system is not controllable. One of the poles located in -4 is impossible to move using state feedback, which limits the achievable speed of the system.

6. The ball and beam process can be controlled by cascaded controllers, where the inner loop concerns control of the beam angle, and the outer loop concerns control of the position of the ball. The block diagram is shown in Figure 1, with notation according to the problem text.

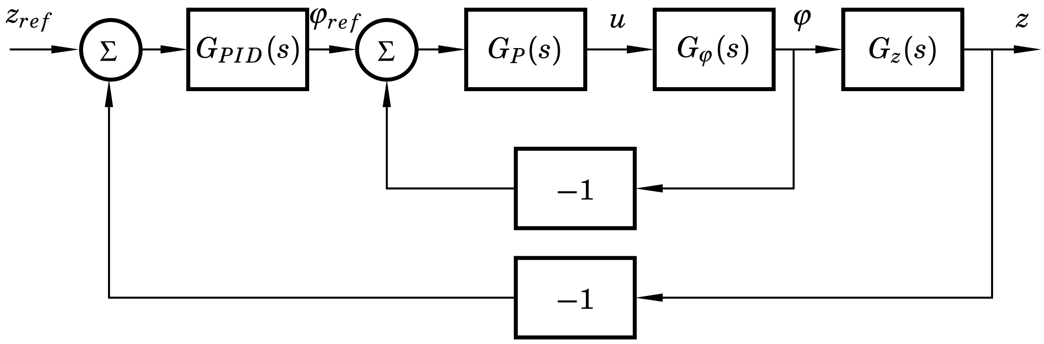

Figure 1 Cascaded controllers in Problem 6.

7. We have to figure out at which gain of the P controller each of the three systems becomes unstable, which is given by the amplitude margin. The amplitude margin is given from where the Nyquist curve crosses the negative real axis, as $-1/x_c$, if x_c is the coordinate for the crossing point. The Nyquist curves give

$$\begin{aligned} A_m^1 &\approx -1/(-0.25) = 4 \\ A_m^2 &\approx -1/(-0.3) = 10/3 \\ A_m^3 &\approx -1/(-0.2) = 5 \end{aligned}$$

which gives a maximum K of $10/3 \approx 3.33$.

8 a. The controller transfer function is given by $G_C(s) = 1$. The transfer function from the reference r to the control error e becomes

$$G_{er}(s) = \frac{1}{1 + G_C(s)G_P(s)} = \frac{1}{1 + \frac{4}{s(s+2)}} = \frac{s(s+2)}{s^2 + 2s + 4}$$

The reference signal is a unit step, i.e., $R(s) = \frac{1}{s}$. Since the system is asymptotically stable, the final value theorem may be used. This gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sG_{er}(s)R(s) = \lim_{s\to 0} s\frac{s(s+2)}{s^2 + 2s + 4}\frac{1}{s} = 0$$

b. We want to reduce the stationary error due to unit ramp disturbances in r from 0.5 to 0.1, i.e. by a factor 5. Thus, we choose a lag filter given by

$$G_k(s) = \frac{s+a}{s+a/M}, \quad M > 1.$$

Since the process has an integrator, and we want the error to decrease by a factor 5, we choose $M = 5$.

We can also compute the stationary error using the final value theorem. The transfer function from r to e with the compensation link included is given by

$$G_{er}(s) = \frac{1}{1 + G_P(s)G_k(s)} = \frac{Ms^3 + (2M + a)s^2 + 2as}{Ms^3 + (2M + a)s^2 + (4M + 2a)s + 4Ma}$$

The conditions for stability of the third order polynomial with $M > 1$ are

$$\begin{aligned}
&2M + a > 0, \\
&4M + 2a > 0, \\
&4Ma > 0, \\
&(2M + a)(4M + 2a) > 4Ma
\end{aligned}$$

which means that the system is asymptotically stable as long as $a > 0$ and $M > 0$. Thus, we can use the final value theorem to compute the stationary error due to a ramp disturbance $R(s) = \frac{1}{s^2}$. We get

$$\begin{aligned}
&\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} sG_{er}(s)R(s) = \\
&= \lim_{s\to 0} s \cdot \frac{Ms^3 + (2M + a)s^2 + 2as}{Ms^3 + (2M + a)s^2 + (4M + 2a)s + 4Ma} \cdot \frac{1}{s^2} = \frac{2a}{4Ma} = \frac{1}{2M}
\end{aligned}$$

We choose $M = 5$, which gives a stationary error of 0.1.

The next step is to determine a. The problem text states that we do not want the filter to give any major change in robustness. We use the rule of thumb $a = 0.1\omega_c$ that guarantees that the phase margin is decreased by at most 6°. ω_c is the crossover frequency for the uncompensated system, which is given by

$$1 = |G_P(i\omega_c)| = \frac{4}{\omega_c\sqrt{\omega_c^2 + 4}} \Rightarrow \omega_c = \sqrt{-2 + \sqrt{20}} = 1.57$$

We thus choose $a = 0.16$.

The lag compensator becomes

$$G_k(s) = \frac{s + a}{s + a/M} = \frac{s + 0.16}{s + 0.031}$$