Second-Order System, Real Poles

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)}$$

Poles:

$$s = -1/T_1, \quad s = -1/T_2$$

Step response in Laplace domain:

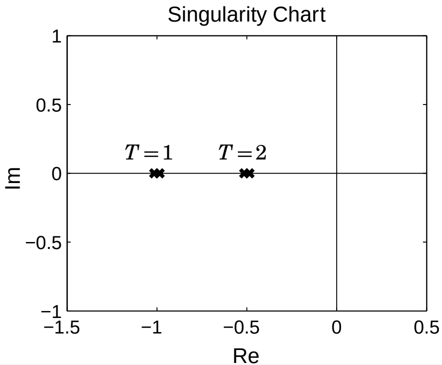
$$Y(s) = G(s)\frac{1}{s} = \frac{K}{s(1+sT_1)(1+sT_2)}$$

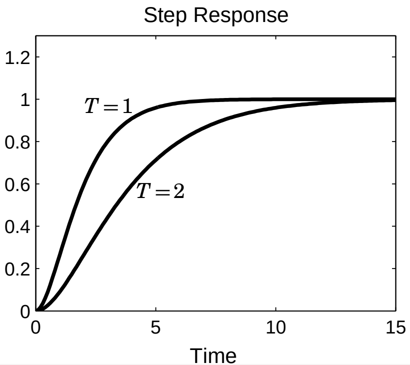
Step response in time domain (transforms 24, 25):

$$y(t) = \begin{cases} K\left(1 - \dfrac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right), & T_1 \neq T_2 \\ K\left(1 - e^{-t/T} - \dfrac{t}{T} e^{-t/T}\right), & T_1 = T_2 = T \end{cases}$$

Second-Order System, Real Poles

- Unbounded response if $T_1 < 0$ or $T_2 < 0$
- From now on assume $T_1, T_2 > 0$
- Pole locations and step responses for $K = 1$ and different values of $T = T_1 = T_2$:





Second-Order System, Real Poles

If the time constants are very different,

$$T_1 \gg T_2,$$

the system essentially behaves as a first-order system with time constant T_1

Second-Order System, Real Poles

Calculation of final value:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{sK}{s(1+sT_1)(1+sT_2)} = K$$

Calculation of initial derivative:

$$\lim_{t\to 0} \dot{y}(t) = \lim_{s\to\infty} s \cdot sY(s) = \lim_{s\to\infty} \frac{s^2K}{s(1+sT_1)(1+sT_2)} = 0$$

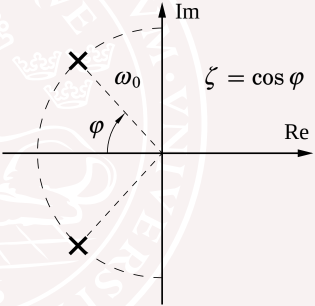
The step response starts smoothly

Second-Order System, Complex Poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad \omega_0 > 0,\ \ 0 < \zeta < 1$$

Poles:

$$s = -\zeta\omega_0 \pm i\sqrt{1-\zeta^2}\,\omega_0$$



- ω_0 – (undamped) natural frequency
- ζ – relative damping

Second-Order System, Complex Poles

Step response in Laplace domain:

$$Y(s) = G(s)\frac{1}{s} = \frac{K\omega_0^2}{s(s^2 + 2\zeta\omega_0 s + \omega_0^2)}$$

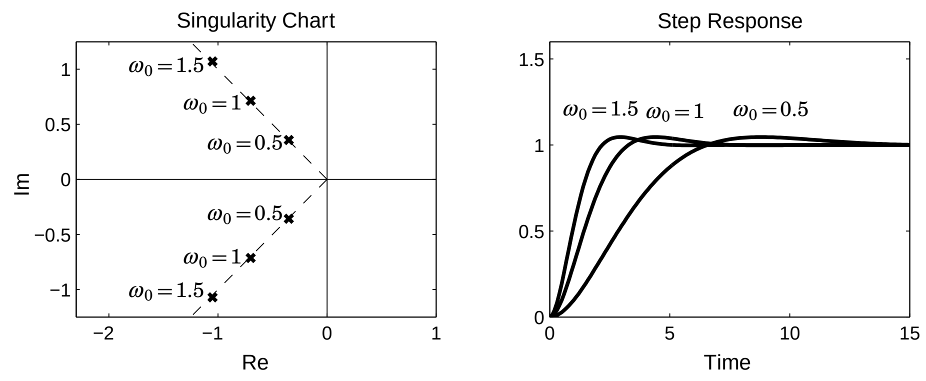
Step response in time domain (transform 28):

$$y(t) = K\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t + \arccos\zeta\right)\right)$$

Damped frequency: $\omega = \omega_0\sqrt{1-\zeta^2}$, period: $T = \dfrac{2\pi}{\omega}$

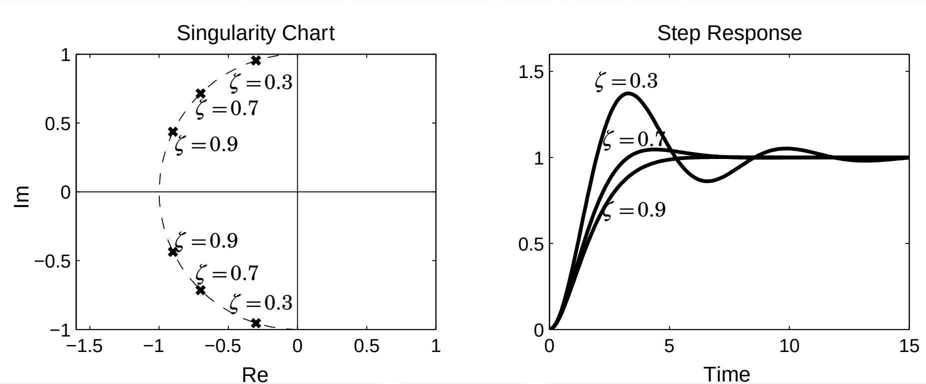
Second-Order System, Complex Poles

Pole locations and step responses for $K = 1$, $\zeta = 0.7$ and different values of ω_0:



Second-Order System, Complex Poles

Pole locations and step responses for $K = 1$, $\omega_0 = 1$ and different values of ζ:



Second-Order System, Complex Poles

Calculation of final value:

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{sK\omega_0^2}{s(s^2 + 2\zeta\omega_0 s + \omega_0^2)} = K$$

Calculation of initial derivative:

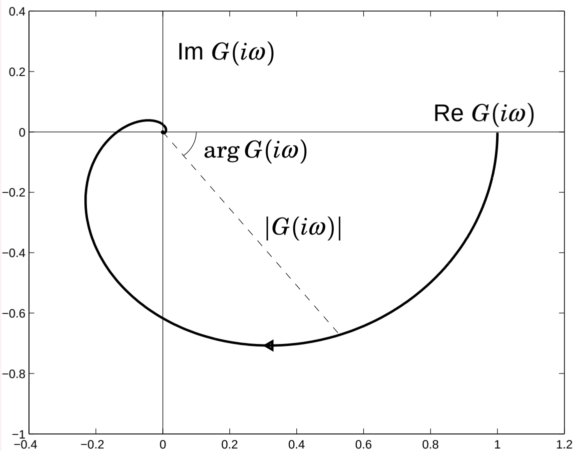
$$\lim_{t\to 0} \dot{y}(t) = \lim_{s\to\infty} s \cdot sY(s) = \lim_{s\to\infty} \frac{s^2 K}{s(s^2 + 2\zeta\omega_0 s + \omega_0^2)} = 0$$

The step response again starts smoothly

The Nyquist Curve

$$u(t) = \sin(\omega t) \Rightarrow y(t) = y_0(t) + a\sin(\omega t + \varphi)$$

$$a = |G(i\omega)|,\ \varphi = \arg G(i\omega)$$



The Bode Plot

Two plots:

- Magnitude: $|G(i\omega)|$
- Phase: $\arg G(i\omega)$

Magnitude and ω drawn in **log** scale, phase in **linear** scale.

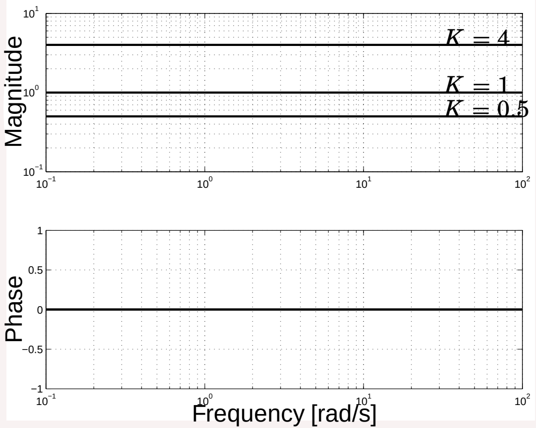
If $G(s)$ can be factored $G(s) = G_1(s)G_2(s)G_3(s)$, then:

$$\log|G(i\omega)| = \log|G_1(i\omega)| + \log|G_2(i\omega)| + \log|G_3(i\omega)|$$
$$\arg G(i\omega) = \arg G_1(i\omega) + \arg G_2(i\omega) + \arg G_3(i\omega)$$

Bode Plot: K

$$\log|G(i\omega)| = \log K$$

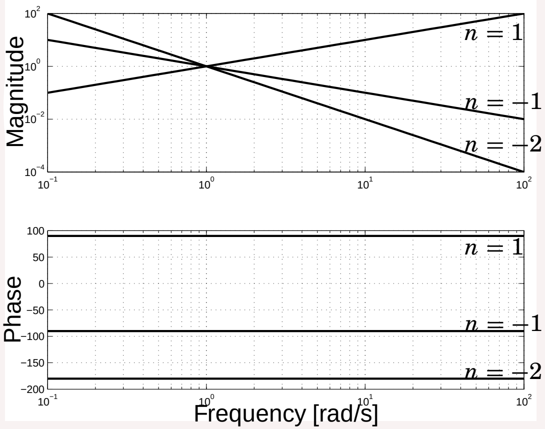
$$\arg G(i\omega) = 0$$



Bode Plot: s^n

$$\log |G(i\omega)| = \log |i\omega|^n = n \log \omega$$

$$\arg G(i\omega) = n \arg(i\omega) = n\frac{\pi}{2}$$



Bode Plot: $(1+sT)^n$

$$\log|G(i\omega)| = n\log\sqrt{1+\omega^2T^2}$$

$$\arg G(i\omega) = n\arg(1+i\omega T) = n\arctan(\omega T)$$

For small values of ω the functions are given by

$$\log|G(i\omega)| \to 0$$

$$\arg G(i\omega) \to 0$$

For large values of ω the functions are given by

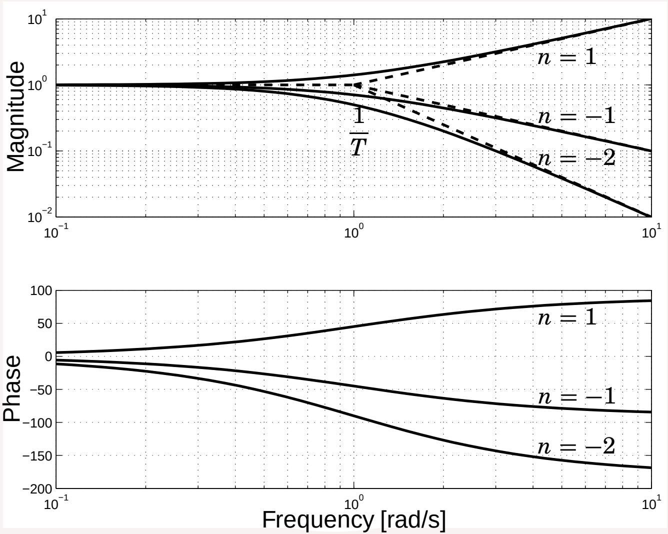
$$\log|G(i\omega)| \to n\log\omega T$$

$$\arg G(i\omega) \to n\frac{\pi}{2}$$

$$\log\omega T = 0$$

This frequency is called the corner frequency and is given by $\omega = 1/T$.

Bode Plot: $(1+sT)^n$



Bode Plot: $(1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$

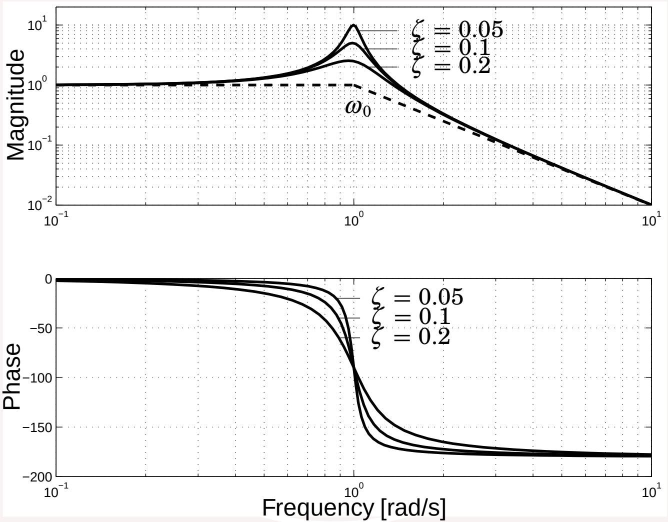
The low-frequency asymptote of this transfer function is given by $G(i\omega) \approx 1$, i.e.

$$\log|G(i\omega)| \to 0$$
$$\arg G(i\omega) \to 0$$

For large ω the high-frequency asymptote is given by $G(i\omega) \approx (i\omega/\omega_0)^{2n} = (-1)^n(\omega/\omega_0)^{2n}$, i.e.

$$\log|G(i\omega)| \to 2n \log\frac{\omega}{\omega_0}$$
$$\arg G(i\omega) \to n\pi$$

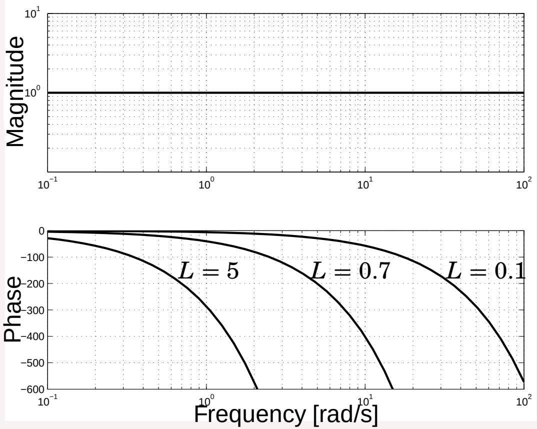
Bode Plot: $(1 + 2\zeta s/\omega_0 + (s/\omega_0)^2)^n$



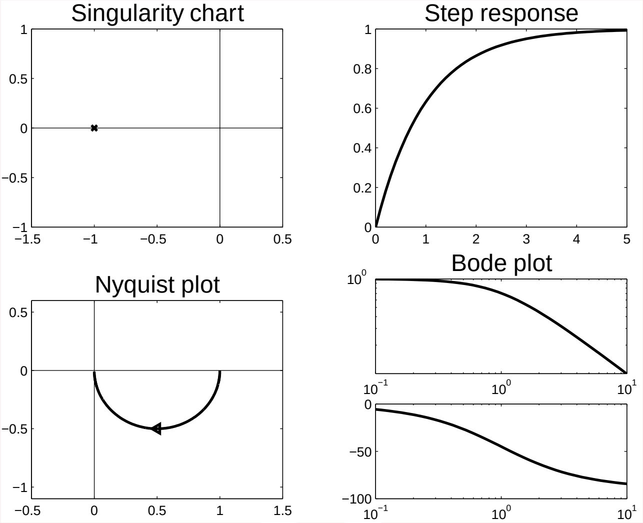
Bode Plot: e^{-sL}

$$\log|G(i\omega)| = \log|e^{-i\omega L}| = 0$$

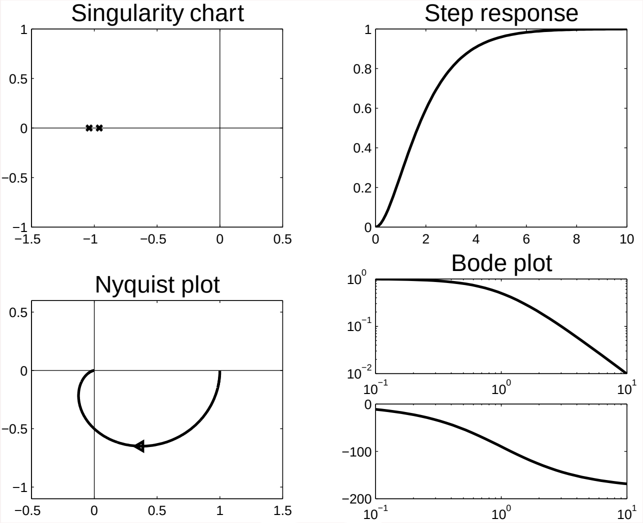
$$\arg G(i\omega) = \arg e^{-i\omega L} = -\omega L$$



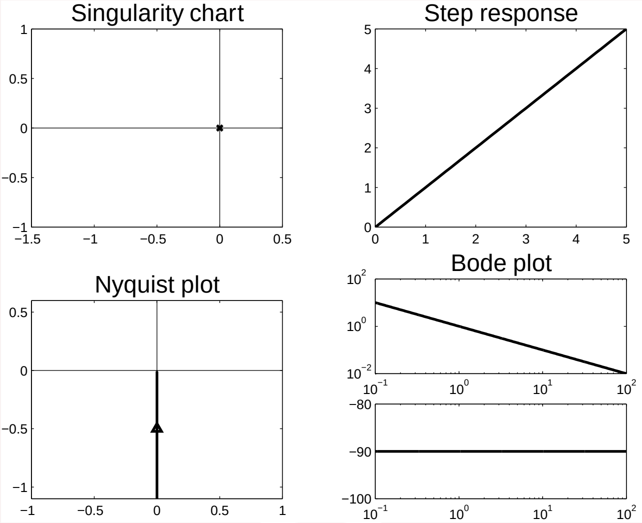
Different representations: $1/(s+1)$



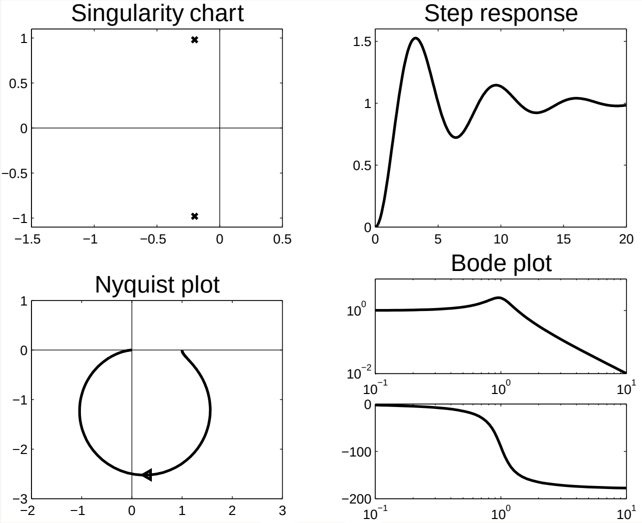
Different representations: $1/(s+1)^2$



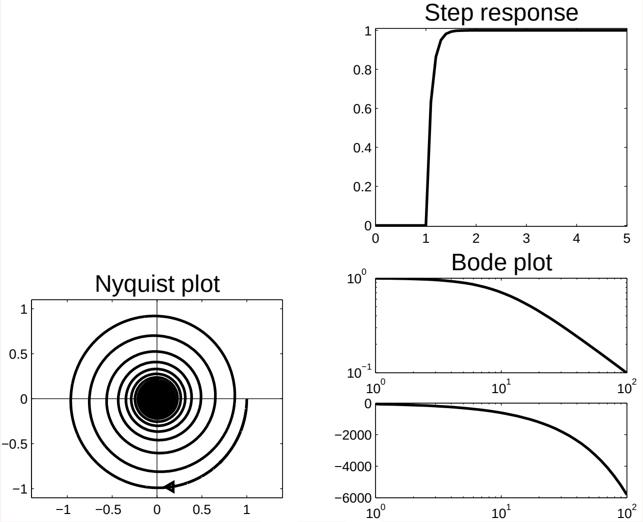
Different representations: $1/s$



Different representations: $1/(s^2 + 0.4s + 1)$



Different representations: $e^{-s}/(0.1s+1)$



Different representations: $(1-s)/((s+0.8)(s+1.2))$