AUTOMATIC CONTROL

Collection of Formulae

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Matrix theory

Notations

Matrix of order $m\,x\,n$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Vector of dimension n

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Transpose

$$B = A^T$$
$$b_{ij} = a_{ji}$$
$$(AB)^T = B^T A^T$$

The matrix is symmetric if $a_{ij} = a_{ji}$.

Determinant

$$\det A = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

If A is of order $2\,x\,2$, then

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

In general

$$\begin{aligned} \det A &= \sum_{i=1}^{n} a_{ij}(-1)^{i+j} \det M_{ij} \\ &= \sum_{j=1}^{n} a_{ij}(-1)^{i+j} \det M_{ij} \end{aligned}$$

where M_{ij} is the matrix one obtains if row i and column j are removed from the matrix A.

Inverse

$$A^{-1}A = AA^{-1} = I \qquad (\det A \neq 0)$$

If A is of order $2\,x\,2$, then

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

In general,

$$A^{-1} = \frac{1}{\det A} C^T$$

where the elements in C are given by

$$c_{ij} = (-1)^{i+j} \det M_{ij}$$

Eigenvalues and eigenvectors

The eigenvalues ($\lambda_i, \quad i = 1, 2, \ldots, n$) and the eigenvectors ($x_i, \quad i = 1, 2, \ldots, n$) are given as the solutions to the equation system

$$Ax = \lambda x$$

which has a solution if

$$\det(\lambda I - A) = \lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_n = 0$$

$\lambda^n + \alpha_1 \lambda^{n-1} + \alpha_2 \lambda^{n-2} + \cdots + \alpha_n$ is called *the characteristic polynomial*. $\det(\lambda I - A) = 0$ is called *the characteristic equation*.

Dynamical systems

State-space equations

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Weighting function

$$y(t) = \int_0^t h(t-\tau)u(\tau)d\tau$$
$$h(t) = Ce^{At}B + D\delta(t)$$

Transfer function

$$Y(s) = G(s)U(s)$$
$$G(s) = C(sI-A)^{-1}B + D = \mathcal{L}\{h(t)\}$$

The denominator of G is the characteristic polynomial to the matrix A.

Frequency response

$$u(t) = \sin\omega t$$
$$y(t) = a\sin(\omega t + \varphi)$$

$$a = |G(i\omega)|$$
$$\varphi = \arg G(i\omega)$$

Linearization

If the nonlinear system

$$\frac{dx}{dt} = f(x,u)$$
$$y = g(x,u)$$

is linearized around a stationary point (x_0, u_0), a change of variables

$$\Delta x = x - x_0$$
$$\Delta u = u - u_0$$
$$\Delta y = y - y_0$$

then gives the linear system

$$\frac{d\Delta x}{dt} = \frac{\partial f}{\partial x}(x_0,u_0)\Delta x + \frac{\partial f}{\partial u}(x_0,u_0)\Delta u$$
$$\Delta y = \frac{\partial g}{\partial x}(x_0,u_0)\Delta x + \frac{\partial g}{\partial u}(x_0,u_0)\Delta u$$

State-space representations

1. Diagonal form

$$\frac{dz}{dt} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} z + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} u$$

$$y = \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_n \end{pmatrix} z + Du$$

2. Observable canonical form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & & 1 \\ -a_n & 0 & 0 & & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix} z + Du$$

3. Controllable canonical form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} z + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

$$y = \begin{pmatrix} b_1 & b_2 & \dots & b_n \end{pmatrix} z + Du$$

The transfer function of the system is

$$\begin{aligned} G(s) &= D + \frac{b_1 s^{n-1} + b_2 s^{n-2} + \cdots + b_n}{s^n + a_1 s^{n-1} + \cdots + a_n} \\ &= D + \frac{\beta_1\gamma_1}{s-\lambda_1} + \frac{\beta_2\gamma_2}{s-\lambda_2} + \cdots + \frac{\beta_n\gamma_n}{s-\lambda_n} \end{aligned}$$

The Laplace transform

Operator lexicon

	Laplace transform $F(s)$	Time function $f(t)$	
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Linearity
2	$F(s+a)$	$e^{-at} f(t)$	Damping
3	$e^{-as} F(s)$	$\begin{cases} f(t-a) & t-a>0 \\ 0 & t-a<0 \end{cases}$	Time delay
4	$\frac{1}{a} F\left(\frac{s}{a}\right) \quad (a>0)$	$f(at)$	Scaling in t-domain
5	$F(as) \quad (a>0)$	$\frac{1}{a} f\left(\frac{t}{a}\right)$	Scaling in s-domain
6	$F_1(s) F_2(s)$	$\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau$	Convolution in t-domain
7	$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F_1(\sigma) F_2(s-\sigma)\, d\sigma$	$f_1(t) f_2(t)$	Convolution in s-domain
8	$sF(s) - f(0)$	$f'(t)$	Differentiation in t-domain
9	$s^2 F(s) - s f(0) - f'(0)$	$f''(t)$	
10	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$	$f^{(n)}(t)$	
11	$\frac{d^n F(s)}{ds^n}$	$(-t)^n f(t)$	Differentiation in s-domain
12	$\frac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$	Integration in t-domain
13	$\int_s^{\infty} F(\sigma)\, d\sigma$	$\frac{f(t)}{t}$	Integration in s-domain
14	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final value theorem
15	$\lim_{s\to\infty} sF(s)$	$\lim_{t\to 0} f(t)$	Initial value theorem

Transform lexicon

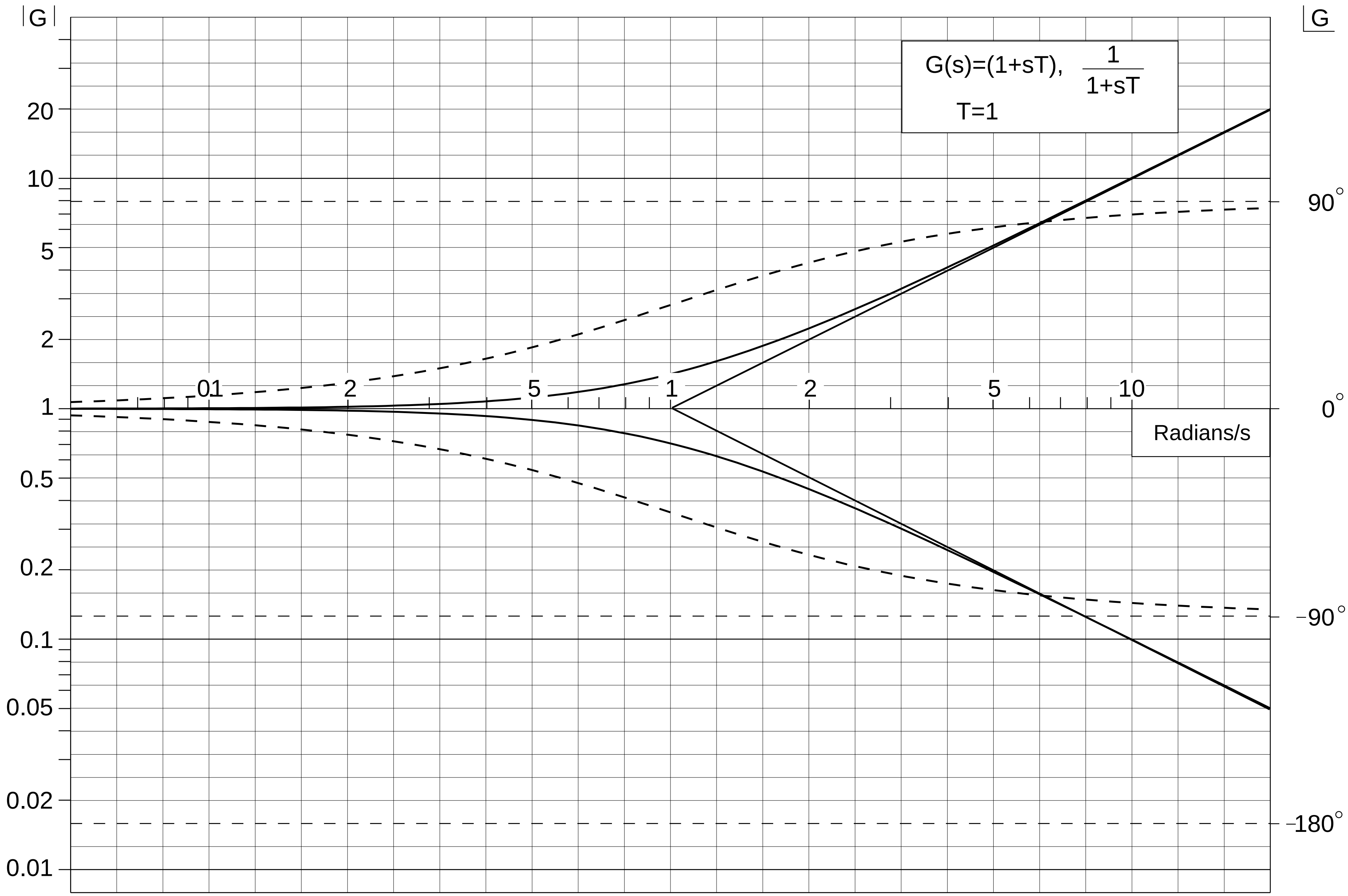
	Laplace transform $F(s)$	Time function $f(t)$	
1	1	$\delta(t)$	Dirac function
2	$\frac{1}{s}$	1	Step function
3	$\frac{1}{s^2}$	t	Ramp function
4	$\frac{1}{s^3}$	$\frac{1}{2}t^2$	Acceleration
5	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	
6	$\frac{1}{s+a}$	e^{-at}	
7	$\frac{1}{(s+a)^2}$	$t \cdot e^{-at}$	
8	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
9	$\frac{1}{1+sT}$	$\frac{1}{T}\, e^{-t/T}$	
10	$\frac{a}{s^2+a^2}$	$\sin at$	
11	$\frac{a}{s^2-a^2}$	$\sinh at$	
12	$\frac{s}{s^2+a^2}$	$\cos at$	
13	$\frac{s}{s^2-a^2}$	$\cosh at$	
14	$\frac{1}{s(s+a)}$	$\frac{1}{a}\left(1-e^{-at}\right)$	
15	$\frac{1}{s(1+sT)}$	$1-e^{-t/T}$	
16	$\frac{1}{(s+a)(s+b)}$	$\frac{e^{-bt}-e^{-at}}{a-b}$	

Transform lexicon, continued

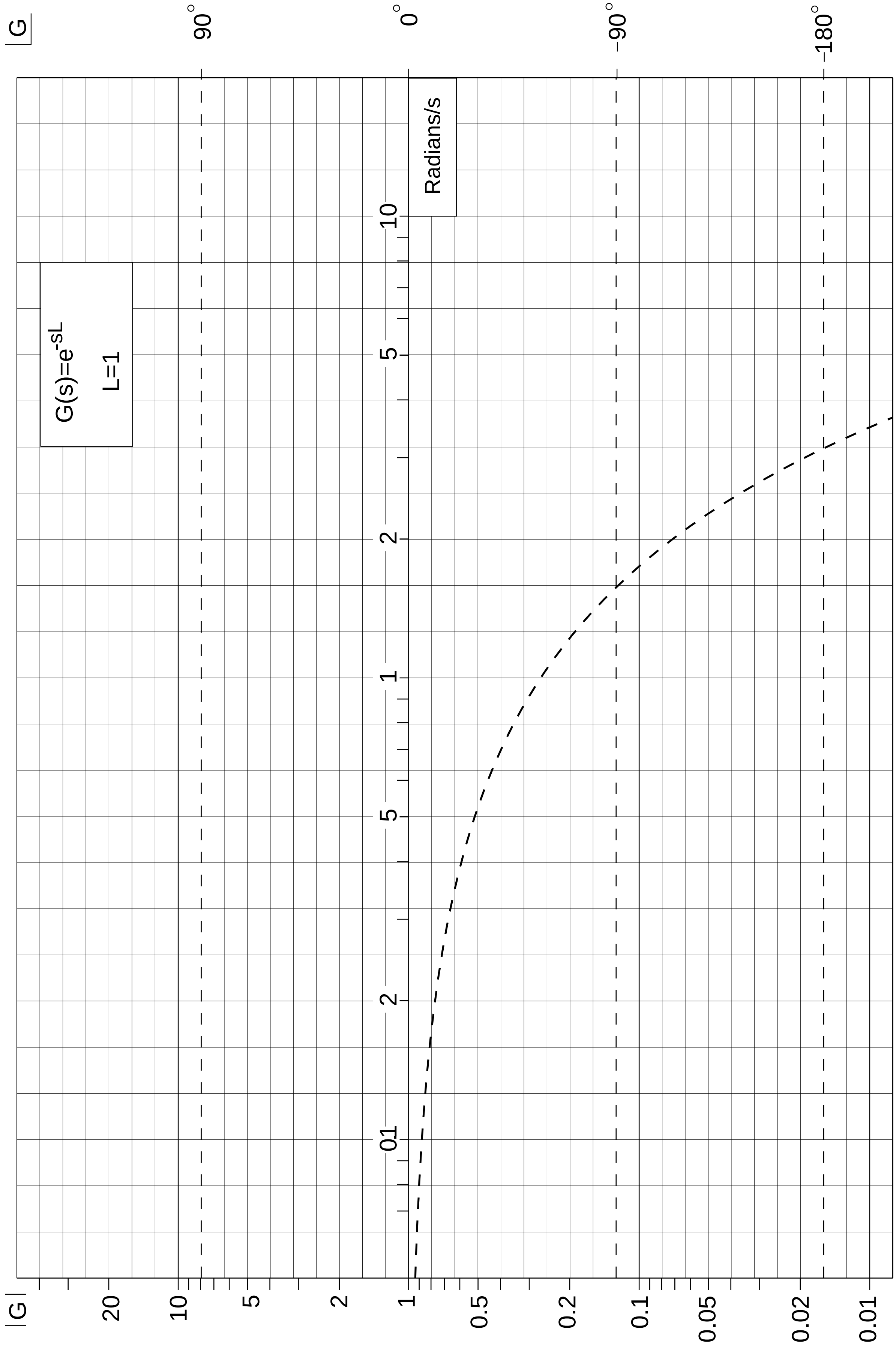
	Laplace transform $F(s)$		Time function $f(t)$
17	$\frac{s}{(s+a)(s+b)}$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
18	$\frac{a}{(s+b)^2+a^2}$		$e^{-bt}\sin at$
19	$\frac{s+b}{(s+b)^2+a^2}$		$e^{-bt}\cos at$
20	$\frac{1}{s^2+2\zeta\omega_0 s+\omega_0^2}$		
		$\zeta=0$	$\frac{1}{\omega_0}\sin\omega_0 t$
		$\zeta<1$	$\frac{1}{\omega_0\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t\right)$
		$\zeta=1$	$te^{-\omega_0 t}$
		$\zeta>1$	$\frac{1}{\omega_0\sqrt{\zeta^2-1}}\,e^{-\zeta\omega_0 t}\sinh\left(\omega_0\sqrt{\zeta^2-1}\,t\right)$
21	$\frac{s}{s^2+2\zeta\omega_0 s+\omega_0^2}$		
	$0\le\tau\le\pi:$	$\zeta<1$	$\frac{1}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t+\tau\right)$
			$\tau=\arctan\frac{\omega_0\sqrt{1-\zeta^2}}{-\zeta\omega_0}$
		$\zeta=0$	$\cos\omega_0 t$
		$\zeta=1$	$(1-\omega_0 t)e^{-\omega_0 t}$
22	$\frac{a}{\left(s^2+a^2\right)(s+b)}$		$\frac{1}{\sqrt{a^2+b^2}}\left(\sin(at-\phi)+e^{-bt}\sin\phi\right)$
			$\phi=\arctan\frac{a}{b}$

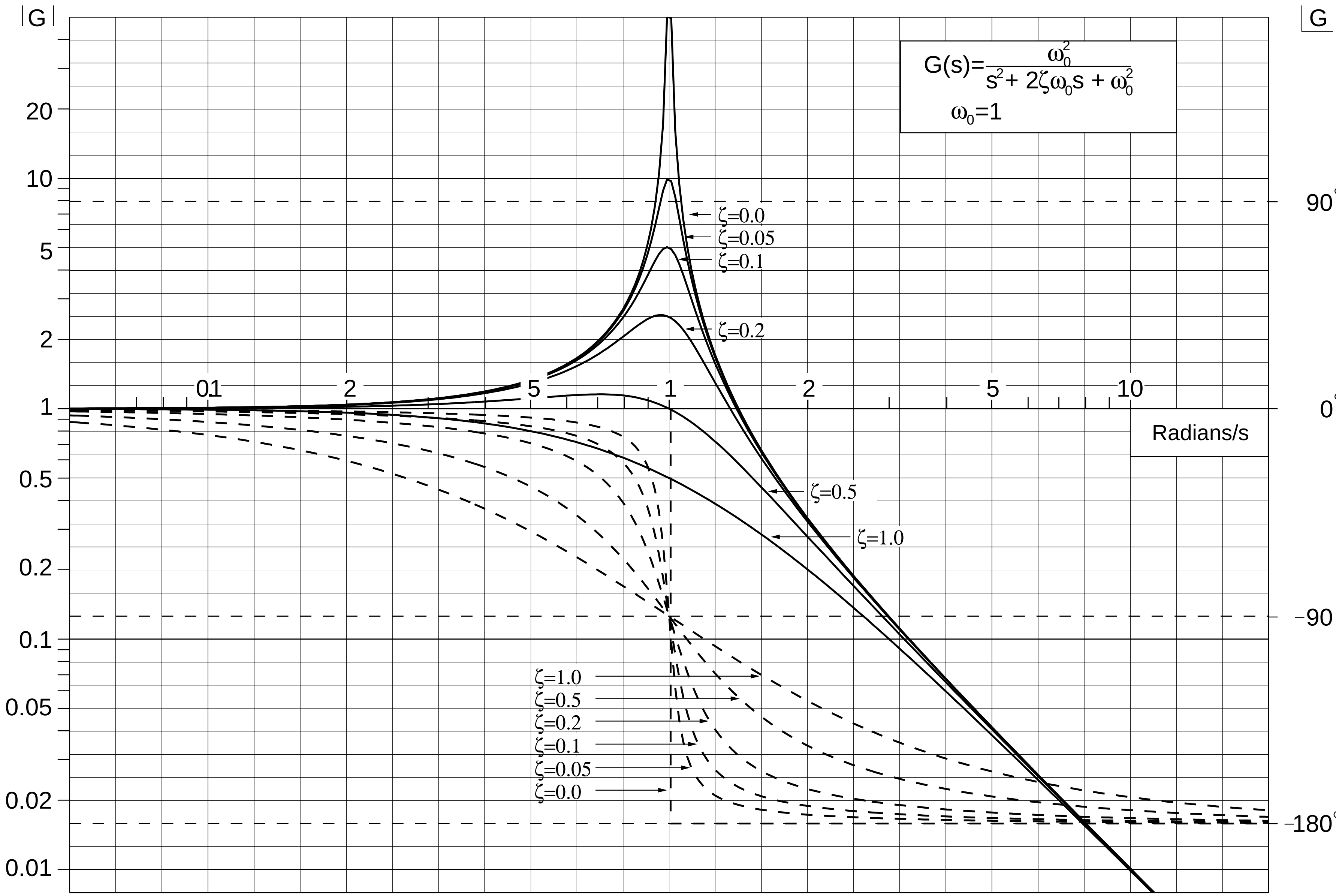
Laplace transform table, continued

	Laplace transform $F(s)$	Time function $f(t)$
23	$\frac{s}{(s^2+a^2)(s+b)}$	$\frac{1}{\sqrt{a^2+b^2}}\left(\cos(at-\phi)-e^{-bt}\cos\phi\right)$ $\phi=\arctan\frac{a}{b}$
24	$\frac{ab}{s(s+a)(s+b)}$	$1+\frac{ae^{-bt}-be^{-at}}{b-a}$
25	$\frac{a^2}{s(s+a)^2}$	$1-(1+at)e^{-at}$
26	$\frac{a}{s^2(s+a)}$	$t-\frac{1}{a}(1-e^{-at})$
27	$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(b-a)(c-a)(b-c)}$
28	$\frac{\omega_0^2}{s(s^2+2\zeta\omega_0 s+\omega_0^2)}$	
	$0<\zeta<1$	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t+\phi\right)$
		$\phi=\arccos\zeta$
	$\zeta=0$	$1-\cos\omega_0 t$
29	$\frac{1}{(s+a)^{n+1}}$	$\frac{1}{n!}t^n e^{-at}$
30	$\frac{s}{(s+a)(s+b)(s+c)}$	$\frac{a(b-c)e^{-at}+b(c-a)e^{-bt}+c(a-b)e^{-ct}}{(b-a)(b-c)(a-c)}$
31	$\frac{as}{(s^2+a^2)^2}$	$\frac{t}{2}\sin at$
32	$\frac{1}{\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}$
33	$\frac{1}{\sqrt{s}}F(\sqrt{s})$	$\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\sigma^2/4t}f(\sigma)\,d\sigma$



Bode plot. Standard curves


G(s)=e^-sL
L=1
Radians/s
|G|
90°
0°
−90°
−180°
20
10
5
2
1
0.5
0.2
0.1
0.05
0.02
0.01
01



Bode plot. Standard curves, continued

Stability

Stability conditions for low-order polynomials

$$
\begin{array}{rl}
s + a_1 & a_1 > 0 \\
s^2 + a_1 s + a_2 & a_1 > 0, \quad a_2 > 0 \\
s^3 + a_1 s^2 + a_2 s + a_3 & a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_1 a_2 > a_3
\end{array}
$$

Routh's algorithm

Consider the polynomial

$$F(s) = a_0 s^n + b_0 s^{n-1} + a_1 s^{n-2} + b_1 s^{n-3} + \cdots$$

Assume that the coefficients a_i, b_i are real and that a_0 is positive. Form the table

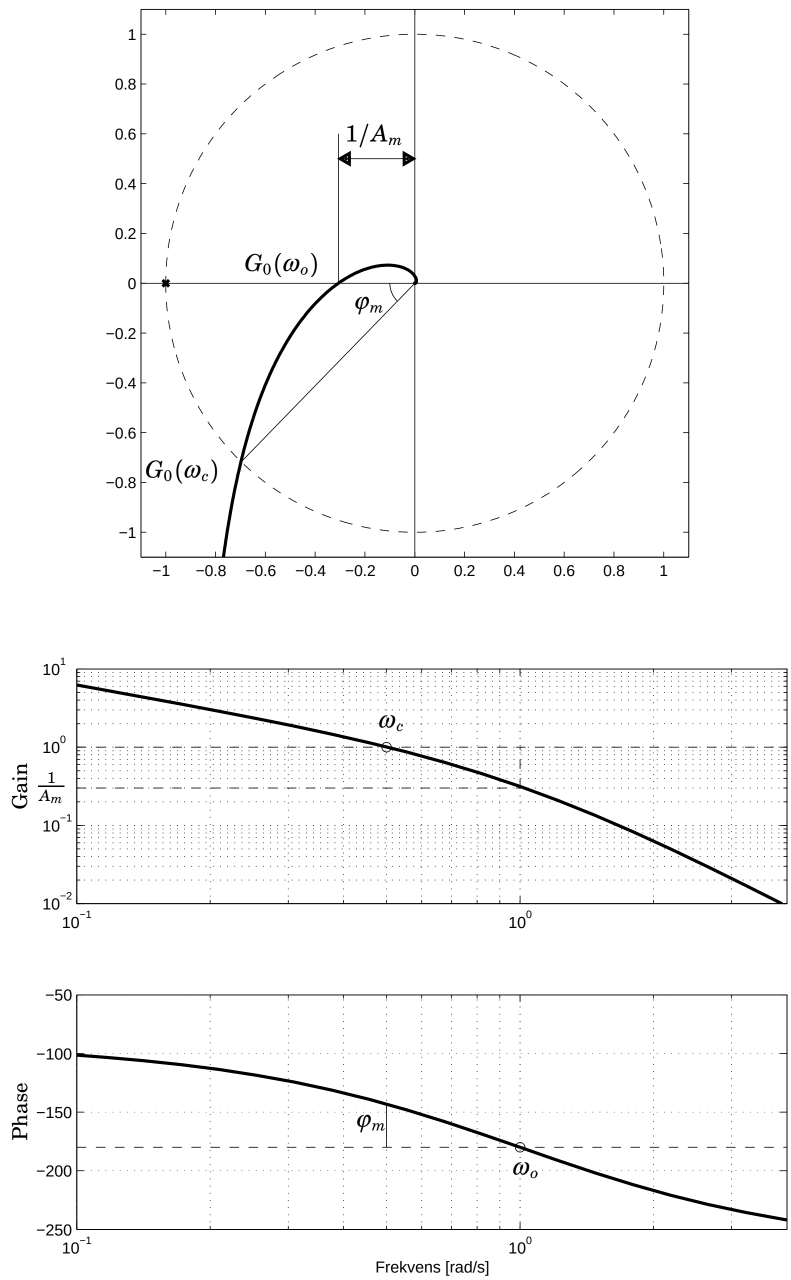
$$
\begin{array}{lll}
a_0 & a_1 & a_2 \ldots \\
b_0 & b_1 & b_2 \ldots \\
c_0 & c_1 & c_2 \ldots \\
d_0 & d_1 & d_2 \ldots \\
\vdots & &
\end{array}
$$

where

$$
\begin{aligned}
c_0 &= a_1 - a_0 b_1 / b_0 \\
c_1 &= a_2 - a_0 b_2 / b_0 \\
&\vdots \\
d_0 &= b_1 - b_0 c_1 / c_0 \\
d_1 &= b_2 - b_0 c_2 / c_0 \\
&\vdots
\end{aligned}
$$

The number of sign changes in the sequence $a_0, b_0, c_0, d_0 \cdots$ equal the number of roots for the polynomial $F(s)$ in the right half plane Re $s > 0$. All the roots of the polynomial $F(s)$ lie in the left half plane if all numbers a_0, b_0, c_0, d_0, $\ldots$ are positive.

Stability margins



Gain margin:

$$A_m = 1/\left|G_0(i\omega_0)\right|$$

Phase margin:

$$\varphi_m = \pi + \arg G_0(i\omega_c)$$

Delay margin:

$$L_m = \varphi_m/\omega_c$$

State feedback and Kalman filtering

State feedback

If the system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx$$

has the control law

$$u = -Lx + \ell_r r$$

then the closed-loop system is given by

$$\frac{dx}{dt} = (A - BL)x + B\ell_r r$$
$$y = Cx$$

Criterion for controllability. The controllable states belong to the linear subspace which is spanned by the columns of the matrix

$$W_s = \begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

A system is controllable if and only if the matrix W_s has rank n.

Kalman filtering

Assume that only the output signal y can be directly measured. Introduce the model

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

The reconstruction error $\tilde{x} = x - \hat{x}$ satisfies

$$\frac{d\tilde{x}}{dt} = (A - KC)\tilde{x}$$

Criterion for observability. The subspace of unobservable states is the null space of the matrix

$$W_o = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

A system is observable if and only if the matrix W_0 has rank n.

Lead-lag compensation

Lag compensator

$$G_K(s) = \frac{s+a}{s+a/M} = M\frac{1+s/a}{1+sM/a} \qquad M > 1$$

The rule of thumb

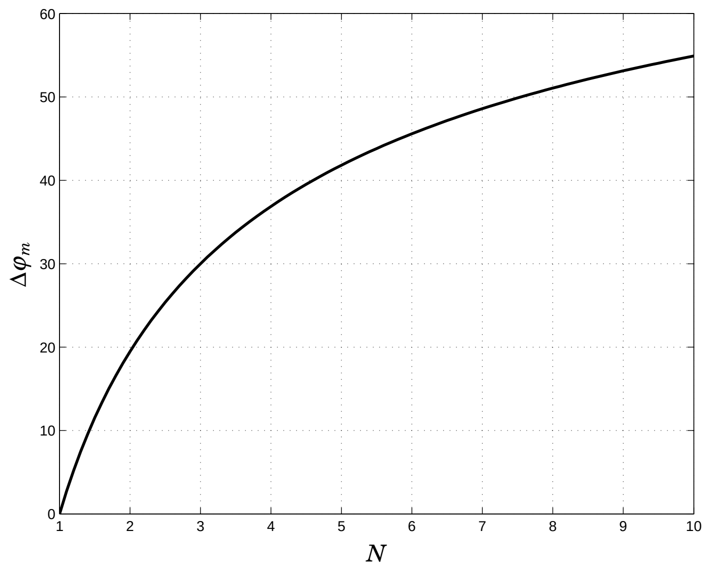
$$a = 0.1\omega_c$$

guarantees that the phase margin is reduced by less than 6°.

Lead compensator

$$G_K(s) = K_K N\frac{s+b}{s+bN} = K_K\frac{1+s/b}{1+s/(bN)} \qquad N > 1$$

The maximum phase advance is given by the figure below:



The peak of the phase curve is located at the frequency

$$\omega = b\sqrt{N}$$

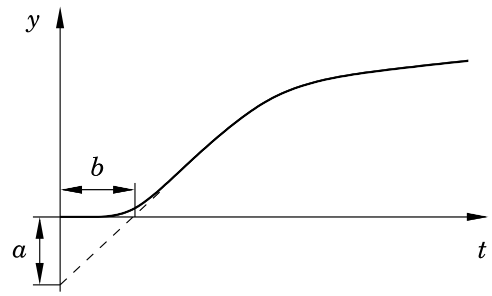
The gain of the compensator at this frequency is

$$K_K\sqrt{N}$$

The Ziegler–Nichols methods

The Ziegler–Nichols step response method

Consider the step response for the *open-loop* system. The tangent is drawn from the point on the step response with the maximal slope. From the intersection of the tangent and the coordinate axes the gain a and time b are found. The PID-parameters are calculated from the table below.



Controller	K	T_i	T_d
P	$1/a$		
PI	$0.9/a$	$3b$	
PID	$1.2/a$	$2b$	$0.5b$

The Ziegler–Nichols frequency method

This method is based on observations of the *closed-loop* system. Outline of the procedure:

1. Disconnect the integral and the derivative part of the PID-controller.
2. Adjust K until the system oscillates with constant amplitude. Denote this value of K as K_0.
3. Measure the period T_0 for the oscillation. The different settings for the controller parameters are given in the table below.

Controller	K	T_i	T_d
P	$0.5K_0$		
PI	$0.45K_0$	$T_0/1.2$	
PID	$0.6K_0$	$T_0/2$	$T_0/8$