1 a. The transfer function from u to y is given by $G(s) = C(sI - A)^{-1}B$.

$$G(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+1 & -1 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

Scoring: 0.5 p for stating the correct algebraic expression and 0.5 p for the final answer.

b. The poles of the system are the eigenvalues of the system matrix A, which are also the zeros of the transfer function denominator polynomial. Solving $(s+1)(s+2) = 0$ shows that the system has one pole in -1 and one pole in -2.

The system is asymptotically stable since the real parts of all its poles are strictly negative.

Scoring: 0.5 p for realizing how to compute the poles, 0.5p for the poles and 0.5 p for the correct conclusion about stability.

2. Breaking the loop at B gives

$$\begin{aligned} B &= Q(P(B+A)+A) = QPB + Q(P+1)A \\ &\Leftrightarrow B(1-QP) = Q(P+1)A \\ &\Leftrightarrow B = \frac{Q(P+1)}{1-QP}A \end{aligned}$$

The transfer function from A to B is hence

$$\frac{Q(P+1)}{1-QP}$$

Scoring: 1 p for an approach involving breaking the loop and writing down an algebraic equation, 1 p for the solution.

3. The cross-over frequency ω_c is the frequency at which the process gain is unity. This happens at $\omega_c \approx 0.07$ rad/s. (Values 0.06 rad/s $\leq \omega_c \leq 0.08$ rad/s qualify as correct answer.)

The $-180°$ phase shift frequency ω_0 is the frequency at which the phase shift is $-180°$. This happens for an ω_0 in the range 0.4 rad/s $\leq \omega_0 \leq 0.5$ rad/s. (Any values within this range qualifies as correct answer).

The amplitude margin A_m is the factor by which the gain at ω_0 needs to be multiplied to reach unity. In the particular Bode plot it is $A_m \approx 1/0.07 \approx 14$. (Any value in the range $1/0.08 \leq A_m \leq 1/0.04$ qualifies as correct answer).

The phase margin φ_m is the phase difference between the phase at ω_c and $-180°$. In the particular Bode plot it is $\varphi_m \approx 35°$. (Any value in the range $30° \leq \varphi_m \leq 45°$ qualifies as correct answer.)

Scoring: 0.5 p per correct definition and corresponding numerical value.

4. By applying the Laplace transform to both sides of the differential equations we obtain the following transfer functions

$$G_1(s) = \frac{1}{s^2 + 0.2s + 1} \qquad G_2(s) = \frac{1}{s + 0.5}$$
$$G_3(s) = \frac{1}{s^2 + 0.8s + 1} \qquad G_4(s) = \frac{2}{4s + 1}$$

$G_1 - G_4$ have all their poles strictly in the left half plane. (For a second order system this is equivalent to all characteristic polynomial coefficients being strictly positive.) Hence $G_1 - G_4$ are asymptotically stable systems. We can therefore eliminate step response D from the candidates.

Step response A is eliminated since none of $G_1 - G_4$ exhibit a time delay.

This leaves B, C, E, and F. Systems G_2 and G_4 have real poles which means their step responses do not oscillate. Candidate step responses for these systems are therefore B and F. We see that the system corresponding to B must be the faster of the two. G_2 has a pole in 0.5, and G_3 has a pole in 0.25. We conclude

$$G_2 \leftrightarrow B$$
$$G_4 \leftrightarrow F$$

This leaves C and E as candidates for G_1 and G_3, both with the structure

$$G(s) = \frac{\omega^2}{s^2 + 2\zeta\omega s + \omega^2}$$

Both G_1 and G_3 have $\omega = 1$ while G_1 has $\zeta = 0.1$ and G_3 has $\zeta = 0.4$. Therefore G_3 is the better damped of the two and must correspond to the step response with less oscillation, i.e. E. In summary

$$G_1 \leftrightarrow C$$
$$G_2 \leftrightarrow B$$
$$G_3 \leftrightarrow E$$
$$G_4 \leftrightarrow F$$

Scoring: Realizing to use transfer function representation gives 0.5 p and writing down the transfer functions gives an additional 0.5 p. Correct reasoning about stability, delay, system order and damping gives 0.5 p each.

5. Inserting the control law into the state update equation results in $\dot{x} = Ax + Bu = Ax - BLx + Bl_r r$. In the Laplace domain, this can be written $sIX = AX - BLX + Bl_r R$ or $(sI - (A - BL))X = Bl_r R$. Combined with the output equation the expression results in

$$Y = C(sI - (A - BL))^{-1} B l_r R$$

The poles of the closed-loop transfer function from R to Y are the solutions of $\det(sI - (A - BL)) = \det(sI - A + BL) = 0$, i.e.

$$\begin{aligned} 0 &= \left| \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} s + l_1 + 1 & l_2 \\ -1 & s+1 \end{pmatrix} \right| \\ &= s^2 + s(l_1 + 2) + (l_1 + l_2 + 1) \end{aligned}$$

This should match the poles of the desired system, defined through

$$0 = (s+1)(s+2) = s^2 + 3s + 2$$

Matching coefficients of the two polynomials in s yields

$$\begin{cases} s^2: & 1 &= 1 \\ s^1: & l_1 + 2 &= 3 \Rightarrow l_1 = 1 \\ s^0: & l_1 + l_2 + 1 &= 2 \Rightarrow l_2 = 0 \end{cases}$$

In stationarity the closed-loop dynamics are governed by

$$\begin{aligned} &\dot{x} = 0 = (A - BL)x + Bl_r r \Leftrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} l_r r \\ &y = Cx \Leftrightarrow y = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

The above expression is equivalent to the linear equation system

$$\begin{cases} 0 &= -2x_1 + l_r r \\ 0 &= x_1 - x_2 \\ y &= x_2 \end{cases}$$

Together with the condition $r = y$, the above system has the unique solution $l_r = 2$. (The third equation gives $x_2 = r$. The second equation then yields $x_1 = x_2 = r$ which inserted into the first equation gives $-2r + l_r r = 0 \Rightarrow l_r = 2$. The sought state feedback control law is therefore

$$u = -x_1 + 2r$$

Comment: It is also possible to obtain l_r by solving $C(sI - (A - BL))^{-1} B l_r = 1$ for $s = 0$.

Scoring: 1 p for the closed-loop characteristic equation, 1 p for the pole placement and 1 p for l_r.

6 a. In stationarity $0 = m\dot{v} = \alpha d \sin\theta - \beta v^2$. Solving for v gives

$$v = \sqrt{\frac{\alpha d \sin\theta}{\beta}}$$

Since $\sin\theta$ takes on all values in $[0, 1]$ when θ traverses $[0, 90°]$, the possible stationary speeds are those in the closed interval

$$\left[0, \sqrt{\frac{\alpha d}{\beta}}\right]$$

Scoring: 0.5 p for realizing the use of $\dot{v} = 0$ in the differential equation, 0.5 p for the expression relating v and θ in stationarity and 0.5 p for the interval of possible stationary speeds.

b. The stationary control signal is found by solving

$$\begin{aligned} 0 &= m\dot{v} = \alpha d \sin\theta_0 - \beta v^2 \\ &= 10 \cdot 4 \sin\theta_0 - 0.2 \cdot 10^2 \\ \Rightarrow \sin\theta_0 &= \frac{1}{2} \end{aligned}$$

Since $0 \leq \theta_0 \leq 90°$ the above equation has the unique solution

$$\theta_0 = 30° = \frac{\pi}{6} \text{ rad}$$

The stationary point of interest is therefore

$$(\theta_0, v_0) = (30°, 10 \text{ m/s})$$

The nonlinear system is

$$\dot{v} = f(v, \theta) = \frac{\alpha d}{m} \sin\theta - \frac{\beta}{m} v^2$$

Differentiating the dynamics with respect to v and θ, respectively, and evaluating the results at the stationary point (θ_0, v_0) yields

$$\begin{aligned} A &= \left.\frac{\partial f}{\partial v}\right|_{(v,\theta)=(v_0,\theta_0)} &= -\frac{2\beta}{m} v_0 = -0.004 \\ B &= \left.\frac{\partial f}{\partial \theta}\right|_{(\theta,v)=(\theta_0),v_0} &= \frac{\alpha d}{m} \cos\theta_0 = 0.02\sqrt{3} \approx 0.035 \end{aligned}$$

Introduction of the new variables $\Delta\theta = \theta - \theta_0$ and $\Delta v = v - v_0$ gives the linearized system

$$\Delta\dot{v} = -0.004\Delta v + 0.02\sqrt{3}\Delta\theta$$

with Laplace domain representation

$$\Delta v = \frac{0.02\sqrt{3}}{s + 0.004}\Delta\theta = \frac{5\sqrt{3}}{250s + 1}\Delta\theta$$

Scoring: 0.5 p for obtaining the stationary point corresponding, 0.5 p for computing the needed derivatives and an additional 0.5 p for evaluating them at the stationary point of interest. 0.5 p for introduction of new variables.

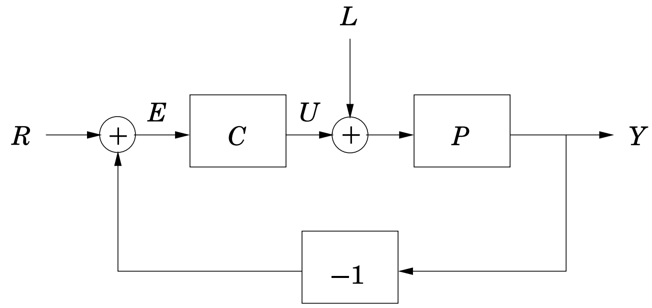

Figure 1 Control system block diagram with components and signals from Problem 6c.

c. See Figure 1.

Scoring: 1 p for a correct solution. 0.5 p if there are 1–2 errors and 0 p if there are more than 2 errors.

d. From the block diagram in the previous subproblem it is possible to directly write down $(-YC + L)P = Y$, assuming $R = 0$. Solving for Y gives the sought transfer function

$$G_{Y,L} = \frac{P}{1 + CP}$$

Scoring: 0.5 p for breaking the loop and writing down the corresponding algebraic equation and 0.5 p for the transfer function.

e. The process has a transfer function with the structure

$$P(s) = \frac{b}{s + a}$$

and the controller is given by $C(s) = K$. Inserting these expressions in the transfer function from the previous subproblem gives

$$G_{Y,L} = \frac{\frac{b}{s+a}}{1 + K = \frac{b}{s+a}} = \frac{b}{s + (a + bK)}$$

The time constant of this transfer function is $T = (a + bK)^{-1}$ [s]. For a given time constant, the desired controller gain is therefore

$$K = \frac{1 - aT}{bT}$$

From subproblem **b.** we have $a = 0.004\ \mathrm{s}^{-1}$ and $b = 0.02\sqrt{3}\ \mathrm{m/s}^2$, which results in

$$K = \frac{1 - 0.004 \cdot 50}{0.02\sqrt{3} \cdot 50} = \frac{0.8}{\sqrt{3}} \approx 0.46\ \mathrm{s/m}$$

If subproblem **b.** was not solved, we instead have $a = 1/100 = 0.01\ \mathrm{s}^{-1}$ and $b = 4/100 = 0.04\ \ \mathrm{m/s}^2$, resulting in $K = 0.25\ \mathrm{s/m}$.

Scoring: 1 p for translating the problem into the correct pole placement equation and 1 p for the correct expression for K. No point deduction is made for incorrect P or $G_{Y,L}$ obtained in previous subproblems as long as they result in a problem of equivalent complexity (otherwise a dection of 0.5 p is made).

7 a. The Laplace transform of the step function is s^{-1}. Differentiation in the time domain is equivalent to multiplication by the Laplace variable s in the Laplace domain. The time derivative of the step response of $G(s)$ therefore has the Laplace transform $G(s)s^{-1}s = G(s)$. The initial value theorem yields

$$\lim_{t\to 0} \frac{d}{dt}\mathcal{L}^{-1}\left(G(s)\frac{1}{s}\right) = \lim_{s\to\infty} sG(s)$$

Applying this result to G_1 and G_2, respectively, yields

$$\lim_{s\to\infty} sG_1(s) = \lim_{s\to\infty} \frac{sb}{s+a} = \lim_{s\to\infty} \frac{s}{s+O(1)}b = \lim_{s\to\infty} \frac{s}{s}b = b$$
$$\lim_{s\to\infty} sG_2(s) = \lim_{s\to\infty} \frac{se}{(s+c)(s+d)} = \lim_{s\to\infty} \frac{se}{s^2+O(s)} = \lim_{s\to\infty} \frac{e}{s} = 0$$

Scoring: 1 p is given for presenting the correct Laplace domain limits and 1 p for correct evaluation of the two limits.

b. A system of order n requires at least n states for its state space representation, but it is always possible to introduce a state space representation with more than n states. Rewriting the system equations

$$\dot{x}_1 = -x_1 + u$$
$$\dot{x}_2 = x_1 - x_2 + 2u$$
$$y = x_1$$

reveals that the state x_2 does not influence x_1 and therefore has no influence on y. The input-output dynamics of the system are given by $\dot{y} = -y + u$ and the system is therefore de facto a first order system with transfer function

$$G(s) = \frac{1}{s+1}$$

It is also possible to arrive at the same conclusion by evaluating $G_{Y,U}(s) = C(sI - A)^{-1}B$, where A, B and C are the system matrices in the problem description

$$C(sI - A)^{-1}B = \begin{pmatrix} 1 & 0 \end{pmatrix} \left(\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \right)^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$
$$= \frac{s+1}{(s+1)(s+1)} = \frac{1}{s+1}$$

The system has a zero at $s = -1$, which cancels one of the two poles at $s = -1$.

Scoring: Analyzing the behavior of the system (either in time or Laplace domain) gives 1 p and the correct conclusion that it is a first order system gives an additional 1 p.

c. The observability matrix of the system is

$$W_o = \begin{pmatrix} C \\ CA \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

Non-observable states x fulfill $W_o x = 0$, i.e.

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The above equality holds whenever $x_1 = 0$, in which case there is no guarantee that the state estimator converges to the correct state.

Comment: The answer "No, because the system is not observable" is also accepted.

Scoring: 0.5 p for computing the observability matrix and an additional 0.5 p for drawing a correct and motivated conclusion.