1. The transfer function from u to y is given by $G(s) = C(sI - A)^{-1}B + D$. We get

$$G(s) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{s+2}{(s+1)(s+2)} = \frac{1}{s+1}$$

2 a. Introducing the state vector $x = \begin{pmatrix} p_1 & \dot{p}_1 & p_2 & \dot{p}_2 \end{pmatrix}^T$ gives

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{m_1} & -\frac{d_1}{m_1} & \frac{k}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{m_2} & 0 & -\frac{k}{m_2} & -\frac{d_2}{m_2} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{k_m}{m_1} \\ 0 \\ 0 \end{pmatrix} u$$

directly from the dynamic model. Since we measure only $p_2 = x_3$, we have

$$y = \begin{pmatrix} 0 & 0 & 1 & 0 \end{pmatrix} x$$

b. The process has one input (u), one output (only p_2 is measured), and four states (p_1, $\dot{p}_1$, p_2 and $\dot{p}_2$).

c. To use state feedback, it is required that all states can be measured. Since this is not the case for the linear servo process, a Kalman filter must be used to estimate the states.

d. The slope at low frequencies in the Bode magnitude plots shows that C is the only controller with integral action. This means that there will be no stationary error due to a load disturbance, which corresponds to step response a. When integral action is introduced, an additional pole is obtained, why this experiment must correspond to poles 2. It can also be seen that step response a is less damped than the other two step responses, which also corresponds well to poles 2, with a damping of $\zeta = 0.5$ for the complex poles, compared to $\zeta = 0.7$ for poles 1 and 3.

Bode diagrams B and C both have no slope for low frequencies, and thus none of the controllers have integral action. The magnitude of B at low frequencies is smaller than the magnitude of A at low frequencies, thus the step response with the greatest stationary error due to the load disturbance, i.e. step response c, corresponds to Bode diagram B. The speed of this step response is also slower than step response b, which gives that the slower poles, i.e. poles 3, belongs to step response c ($\omega = 14$ for poles 1 and $\omega = 5$ for poles 3).

Concluding this, we get

A-b-1 **B-c-3** **C-a-2**

3 a. The cross-over frequency is given by

$$\pi + \arg(KG_P(i\omega_c)) = \pi - \frac{\pi}{2} - 2\arctan(\omega_c) = \frac{45\pi}{180} \Rightarrow \omega_c = \tan\frac{\pi}{8} \approx 0.41 \text{ rad/s}$$

To get the phase margin 45° at the cross-over frequency, we must have

$$|G_R(i\omega_c)G_P(i\omega_c)| = \frac{2K}{\omega_c(1+\omega_c^2)} = 1 \Rightarrow K = \frac{\omega_c(1+\omega_c^2)}{2} \approx 0.24$$

b. Block diagram calculations give:

$$
\begin{aligned}
i)\ G_{yr}(s) &= \frac{G_R(s)G_P(s)}{1+G_R(s)G_P(s)} = \frac{2K}{s^3+2s^2+s+2K} = \frac{0.48}{s^3+2s^2+s+0.48} \\
ii)\ G_{yn}(s) &= \frac{1}{1+G_R(s)G_P(s)} = \frac{s(s+1)^2}{s^3+2s^2+s+2K} = \frac{s(s+1)^2}{s^3+2s^2+s+0.48} \\
iii)\ G_{yd}(s) &= \frac{G_P(s)}{1+G_R(s)G_P(s)} = \frac{2}{s^3+2s^2+s+2K} = \frac{2}{s^3+2s^2+s+0.48} \\
iv)\ G_{ur}(s) &= \frac{G_R(s)}{1+G_R(s)G_P(s)} = \frac{Ks(s+1)^2}{s^3+2s^2+s+2K} = \frac{0.24s(s+1)^2}{s^3+2s^2+s+0.48}
\end{aligned}
$$

c. The maximum delay that can be tolerated is given by the delay margin

$$L_m = \phi_m/\omega_c$$

Using the controller in a), we have $\phi_m = 45^\circ = \pi/4$ and $\omega_c = 0.41$ rad/s. This gives the delay margin

$$L_m = \frac{\pi/4}{0.41} \approx 1.9 \text{ s}$$

4. To compensate for disturbances before they affect the control error is known as *feedforward*. A block diagram of the system is shown in figure 1.

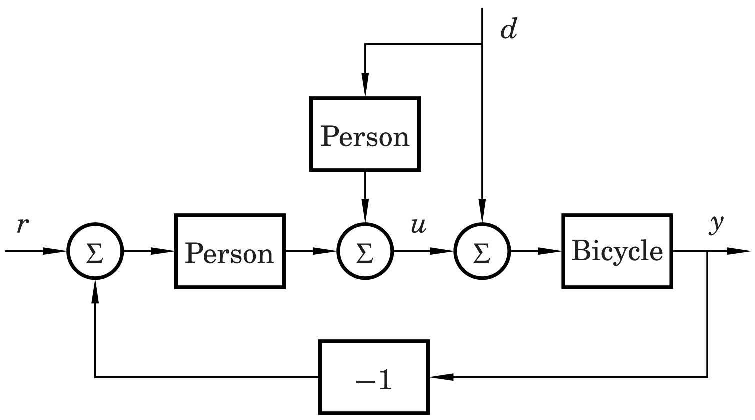

Figure 1 Blockdiagram for problem 4, where y is the speed, r is the reference speed, u is pedaling/braking, and d is the speed disturbance due to the slope in the road.

5 a. The poles of the closed-loop system can be placed arbitrarily if the process is controllable. The controllability matrix is given by

$$W_c = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

The controllability matrix has full rank ($\det(W_c) \neq 0$), and thus the double tank process is controllable, and the closed-loop poles can be placed arbitrarily using state feedback.

b. The closed-loop system is given by

$$\begin{aligned} \dot{x} &= Ax + B(-Lx + l_r r) = (A - BL)x + Bl_r r \\ y &= Cx \end{aligned}$$

Inserting the given values of L and l_r gives

$$\begin{aligned} \dot{x} &= \left(\begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \right) x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot 6r = \begin{pmatrix} -3 & -3 \\ 1 & -1 \end{pmatrix} x + \begin{pmatrix} 6 \\ 0 \end{pmatrix} r \\ y &= \begin{pmatrix} 0 & 1 \end{pmatrix} x \end{aligned}$$

c. $u = -Lx = -2x_1 - 3x_2 = -2x_1 - 3y$. So $Q = -2$ and $W = -3$.

d. The transfer function from reference to output is given by $G(s) = C(sI - A + BL)^{-1}Bl_r$. We get

$$G(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+3 & 3 \\ -1 & s+1 \end{pmatrix}^{-1} \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \frac{6}{s^2 + 4s + 6}$$

Since the system is asymptotically stable (poles in $s = -2 \pm \sqrt{2}i$), the final value theorem can be used. We get

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)R(s)$$

With $R(s) = \frac{1}{s}$ (a step reference), we get

$$\lim_{s\to 0} sG(s)\frac{1}{s} = G(0) = \frac{6}{6} = 1$$

This means that $r = y$ in stationarity, i.e. the stationary error is equal to zero.

e. In the previous subproblem, it was concluded that a reference step did not give any stationary error. However, if there are load disturbances acting on the system, this will result in a stationary error, which is a motivation for including integral action in the controller.

6. The Nyquist curve intersects the negative real axis three times, at approximately -0.5, -1, and -3.5. For the intersection at -0.5, it can be seen that we have to have $0 < K < 1/0.5$ in order for the Nyquist curve to not encircle -1. For the intersection at -1, we see that we have to have $1 < K < 2$. Finally, for the intersection at -3.5 we get $0 < K < 1/3.5$.

Consequently, the system is stable for

$$0 < K < 0.29$$

as well as for

$$1 < K < 2$$

according to the Nyquist theorem.

7 a. Both the lead and the lag compensation links can be described by the transfer function

$$G(s) = K\frac{s+a_1}{s+b_1}$$

where $b_1 > a_1$ if it is a lead compensator, and $a_1 > b_1$ if it is a lag filter. a_1 is the location of the zero, where the Bode amplitude increases, and b_1 is the location of the pole, where the Bode amplitude decreases. Bode diagram A and D can not be lead or lag compensators (with finite parameters), since the slope of the asymptotes in the magnitude plot only changes once. In Bode diagram B, the frequency of the pole is greater than the frequency of the zero, thus this corresponds to a lead compensator. In Bode diagram C, the frequency of the zero is greater than the frequency of the pole, thus this corresponds to a lag compensator.

b. A lead compensator can be used to improve the speed of a system (increase the cutoff frequency) or to improve robustness (increase the phase margin).

c. When $N \to \infty$, the lead compensator becomes

$$G_K = K_K(1 + \frac{s}{b})$$

i.e. a PD controller. The controller parameters become $K = K_K$, $T_d = 1/b$ ($T_i = \infty$) for both the parallel and the series form of the PID controller.