**1.**

**a.** The Nernst equation for ion [i] is given by

$$E_i = \frac{\mathrm{RT}}{\mathrm{zF}} \ln\left(\frac{C_{out,i}}{C_{in,i}}\right)$$

where $z$ - valence charge, $C_{out}$ the ion concentration outside the cell, $C_{in}$ the ion concentration inside the cell, $R$ - thermodynamic gas constant, $F$ - Faraday constant and $T$ - temperature in Kelvin.

Given $R = 8.31447$ [J/mol·K], $T = 273+25$ [K] and $F = 9.648534 \cdot 10^4$ [C/mol] then $RT/F = 0.0257$ [V] or 25.7 [mV].

Using the Nernst equation with the given values of the inner/external concentrations as well as the valence charge results in $E_1 = 23$, $E_2 = -72$ and $E_3 = -12$ [mV].

**b.** A decrease in $T$ results in the coefficient $RT/F$ being smaller. Therefore a decrease in $T$ results in a decrease in $E_1$ and an increase in $E_2$ and $E_3$.

**2.**

**a.** The solution of $\dot{C} = -0.8C$ is given by $C(t) = C_0 e^{-0.8t}$. Solve $0.4C_0 = C_0 e^{-0.8T}$ for $T$ to get the time.

$$\begin{aligned} 0.4C_0 &= C_0 e^{-0.8T} \\ 0.4 &= e^{-0.8T} \\ \ln(0.4) &= -0.8T \\ T &= \frac{\ln(0.4)}{-0.8} = 1.14\text{hours} \approx 69\text{minutes.} \end{aligned}$$

**b.** The kinetics of the system is given by

$$\dot{C} = -0.8C + \frac{u}{V}$$

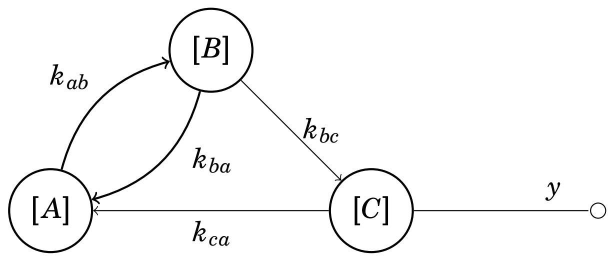
The transfer function of the system is

$$C(s) = \frac{1}{V}\frac{1}{s+0.8}$$

The static gain is $C(0) = 1/(0.8V)$. And the constant dose that achieves the steady state is given by $u = C_{opt}/C(0)$.

**3.**

**a.**




or use $(a,\ V_a)$, and similarly, in each compartment.

**b.** Let $x = [a\ b\ c]^T$. Then the system can be written as

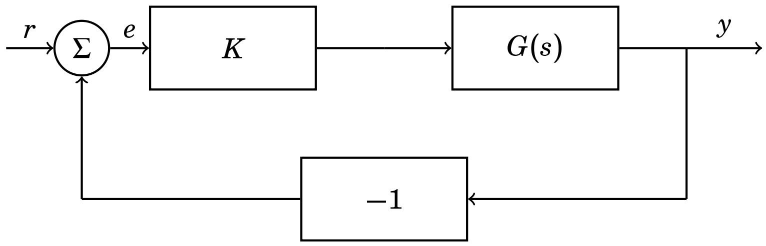
$$\dot{x} = \begin{bmatrix} -k_{ab} & k_{ba} & k_{ca} \\ k_{ab} & -(k_{ba}+k_{bc}) & 0 \\ 0 & k_{bc} & -k_{ca} \end{bmatrix}$$
$$y = [0 \quad 0 \quad 1/V_c]\, x.$$

**c.** It is easy to check that $\dot{a} + \dot{b} + \dot{c} = 0$. Thus $a(t) + b(t) + c(t) = q$, where $q$ is some constant. Using the initial conditions we find that $q = p$.

**4.**

**a.** From the Nyquist plot it is clear that the amplitude margin is $A_m = 1/2$, and thus the larges $K$ is 1/2.

**b.**




**c.** Inserting $u = -Lx + l_r r$ gives $\dot{x} = (A - BL)X + Bl_r r$. The Laplace transform then gives $X(s) = (sI - A + BL)^{-1} Bl_r R(S)$. Thus the transfer function can then be found,

$$Y(s) = \underbrace{C(sI - A + BL)^{-1} Bl_r}_{G(S)} R(S)$$

**d.** Calculations gives

$$|sI - A + BL| = \det \begin{bmatrix} s - 1 + l_1 & l_2 \\ -2 & s - 1 \end{bmatrix} = s^2 + (l_1 - 2)s + 1 - l_1 + 2l_2$$

This is to be compared with $(s+1)^2 = s^2 + 2s + 1$. Identification of coefficients gives $l_1 = 4$ and $l_2 = 2$.

**5.**

**a.** $\dot{x} = 0$ for $x = 0$ or $y = \alpha/\beta$ assuming $x \neq 0$. Likewise for $\dot{y}$ we have that $y = 0$ or $x = \gamma/\delta$ assuming $y \neq 0$. Thus the two stationay points that makes $\dot{x} = 0$ and $\dot{y} = 0$ is $(0, 0)$ and $(\gamma/\delta, \alpha/\beta)$.

**b.** Let

$$f = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}.$$

Then

$$\frac{\partial f}{\partial x}(x, y) = \begin{bmatrix} \alpha - \beta y \\ \delta y \end{bmatrix}, \quad \frac{\partial f}{\partial y}(x, y) = \begin{bmatrix} -\beta x \\ \delta x - \gamma \end{bmatrix}.$$

In the point of interest we have

$$\frac{\partial f}{\partial x}(x^0, y^0) = \begin{bmatrix} 0 \\ \delta\frac{\alpha}{\beta} \end{bmatrix}, \quad \frac{\partial f}{\partial y}(x^0, y^0) = \begin{bmatrix} -\beta\frac{\gamma}{\delta} \\ 0 \end{bmatrix}.$$

Let $\Delta x = x - x^0$ and $\Delta y = y - y^0$. Then we have that

$$\begin{bmatrix} \Delta\dot{x} \\ \Delta\dot{y} \end{bmatrix} = \begin{bmatrix} 0 & -\beta\frac{\gamma}{\delta} \\ \delta\frac{\alpha}{\beta} & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}.$$

**c.** The poles of the systems are given by the solution to $s^2 + \alpha\gamma$. Thus the poles are given by $\pm i\sqrt{\alpha\gamma}$. The poles are on the imaginary axis and the linearized system is marginally stable.

**6.**

**a.** Assuming steady state, the output is given by $y(t) = |G(3i)|\sin(3\omega t + \arg(G(3i)))$.

$$|G(3i)| = \frac{2}{\sqrt{9+9}} = \frac{2}{\sqrt{18}} = \frac{\sqrt{2}}{3}$$
$$\arg(G(3i)) = -\arctan(1) = -\pi/4$$

It is also possible to use the method in the next subproblem.

**b.** We have that $Y(s) = G(s)U(s) = 2/(s(s+3))$ Using the collection of formulae we find that

$$y(t) = 2/3 \cdot (1 - e^{-3t})$$

**c.** The output of the first system can be found by

$$y_1(t) = \int_0^t h(t-\tau)u_2(\tau)\,\mathrm{d}\tau,$$

where $h$ is the impulse response of the linear system. Similarly for the second system

$$\begin{aligned} y_2(t) &= \int_0^t h(t-\tau)u_2(\tau)\,\mathrm{d}\tau = \int_0^t h(t-\tau)2u_1(\tau)\,\mathrm{d}\tau \\ &= 2\int_0^t h(t-\tau)u_1(\tau)\,\mathrm{d}\tau = 2y_1(t) \end{aligned}$$

System B can not be linear as the responses from two different step sizes can not be related only by a positive scaling factor.

**7.** a - ii, b - iv, c- i, d-iii.

The step response in (a) have static gain 0.5.

The step response in (b) have a time delay.

The step response in (c) have static gain 1, but no time delay.

The step response in (d) is unstable (and have complex poles).

Tabell 1: Drug data for problem 1

| Substrate concentration $[S]$ [units] | Reaction rate $V$ [units/days] |
|---|---|
| 0.5 | 0.12 |
| 1 | 0.27 |
| 2 | 0.33 |
| 4 | 0.58 |

**8.** Rewrite relationship as

$$\frac{1}{V} = \frac{K_m}{V_{max}} \frac{1}{[S]} + \frac{1}{V_{max}}.$$

Now, the parameters $K_m/V_{max}$ and $1/V_{max}$ may be estimated as follows: Let the regressor matrix be

$$\Phi = \begin{pmatrix} 1 & 1/[S]_1 \\ 1 & 1/[S]_1 \\ 1 & 1/[S]_3 \\ 1 & 1/[S]_4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0.5 \\ 1 & 0.25 \end{pmatrix}$$

where $[S]_i$ is the i-th value of $[S]$ in Table 1. Denote

$$y = \begin{pmatrix} 1/V_1 & 1/V_2 & 1/V_3 & 1/V_4 \end{pmatrix}^T = \begin{pmatrix} 8.33 & 3.70 & 3.03 & 1.73 \end{pmatrix}^T$$

where $V_i$ is the i-th value of $V$ is Table 1.
The least-squares solution is then

$$\begin{pmatrix} \frac{1}{\hat{V}_{max}} \\ \frac{\hat{K_m}}{V_{max}} \end{pmatrix} = \left(\Phi^T \Phi\right)^{-1} \Phi^T y = \begin{pmatrix} 0.77 \\ 3.66 \end{pmatrix}$$

which results in $\hat{V}_{max} = 1.30$ and $\hat{K}_m = 4.77$.

**Good Luck!**