Last week

- Argument principle
- General Nyquist criterion:
  \[ P_{UNSTABLE}^{CLOSED} - P_{UNSTABLE}^{OPEN} = \text{CW encirclements of -1} \]
- Bode’s relations between gain and phase
Design Tradeoffs

Sensitivity Conservation Law

Linearization around trajectory

- Example: Stabilization of inverted pendulum
Architecture with Two Degrees of Freedom

Ingredients:
- Controller: feedback $C$, feedforward $F$
- Load disturbance $d$: Drives the system from desired state
- Process: transfer function $P$
- Measurement noise $n$: Corrupts information about $x$
- Process variable $x$ should follow reference $r$
Typical Requirements

A controller should

A: Reduce effects of load disturbances
B: Not inject too much measurement noise into the system
C: Make the closed loop insensitive to variations in the process
D: Make output follow command signals well

Systems with two degrees of freedom (2DOF)

- Design feedback $C$ for A, B and C
- Then design feed-forward $F$ to handle D

Systems with error feedback ($F = 1$) do not allow this separation of responses to command signal and disturbances.
The Gangs of Four and Seven

\[ X = \frac{P}{1 + PC} D - \frac{PC}{1 + PC} N + \frac{PCF}{1 + PC} R \]

\[ Y = \frac{P}{1 + PC} D + \frac{1}{1 + PC} N + \frac{PCF}{1 + PC} R \]

\[ E = -\frac{P}{1 + PC} D - \frac{1}{1 + PC} N + \frac{F}{1 + PC} R \]

\[ U = -\frac{PC}{1 + PC} D - \frac{C}{1 + PC} N + \frac{CF}{1 + PC} R \]
Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four GoF)

\[
\frac{PC}{1 + PC} \quad \frac{P}{1 + PC} \quad \frac{C}{1 + PC} \quad \frac{1}{1 + PC}
\]

- The system with a controller having two degrees of freedom is characterized by *seven* transfer function (The Gang of Seven GoS)

\[
\frac{PCF}{1 + PC} \quad \frac{CF}{1 + PC} \quad \frac{F}{1 + PC}
\]

- To fully understand a system it is necessary to look at *all* transfer functions

- It may be strongly misleading to only show properties of a few systems for example the response of the output to command signals, a common omission in literature.
Gain curves of the Gang of Four for a heat conduction process with I (dash-dotted), PI (dashed) and PID (full) controllers.

One plot like this gives a good overview of performance and robustness!
One Way to Show All Responses

![Graph showing step responses and measurement noise](image)

**Automatic Control LTH, 2016**

**FRT130 Control Theory, Lecture 3**
Effect of small process changes on $T = \frac{PC}{1 + PC}$

$$\frac{dT}{dP} = \frac{C}{(1 + PC)^2} = \frac{ST}{P}, \quad \frac{dT}{T} = S \frac{dP}{P}$$

**Robustness:** Small relative impact of relative process variations when $S$ is small

How much can the process be changed without making the closed loop unstable?
Robustness against large process variations

Closed loop stability with \( P(s) + \Delta P(s) \) is guaranteed if nominal loop is stable and

\[
|C\Delta P| < |1 + PC|
\]

\( \iff \) \[
\left| \frac{\Delta P}{P} \right| < \left| \frac{1 + PC}{PC} \right| = \frac{1}{|T|}
\]

**Robustness:** Large variations permitted when \( T \) is small

Note \( S + T = 1 \).
Design Tradeoffs

**Sensitivity Conservation Law**

Linearization around trajectory
- Example: Stabilization of inverted pendulum
Effect of Feedback on Disturbances

\[ r = 0 \]
\[ \Sigma \]
\[ e \rightarrow C \rightarrow u \rightarrow \Sigma \]
\[ x \rightarrow P \rightarrow y \]

Output without control: \[ Y_{ol} = N + PD \]

Output with feedback control: \[ Y_{cl} = \frac{1}{1 + PC}(N + PD) = SY_{ol} \]

The sensitivity function \( S = 1/(1 + PC) \) tells how feedback influences the effect of disturbances: Disturbances with frequencies such that \( |S(i\omega)| < 1 \) are reduced by feedback, disturbances with frequencies such that \( |S(i\omega)| > 1 \) are amplified by feedback.
Assessment of Disturbance Reduction

We have

\[ Y_{cl} = SY_{ol}(t), \quad S(s) = \frac{1}{1 + P(s)C(s)} \]

- Feedback attenuates disturbances when \(|S(i\omega)| < 1\)
- Feedback amplifies disturbances when \(|S(i\omega)| > 1\)
- The sensitivity crossover frequency \(\omega_{sc} (|S(i\omega_{sc})| = 1)\) is an important parameter, (there may be many values)
The Water Bed Effect - Bode’s Integral

If the closed loop is stable and $P(s)C(s)$ has relative degree $\geq 2$:

$$\int_{0}^{\infty} \log |S(i\omega)| \, d\omega = \pi \sum_{p_k \in \text{RHP}} \text{Re} \, p_k$$

The sensitivity can be decreased at one frequency at the cost of increasing it at another frequency. Feedback design is a trade-off!
Design Tradeoffs

Sensitivity Conservation Law

Linearization around trajectory
  -Example: Stabilization of inverted pendulum
Linearization around a trajectory

Let \((x_0(t), u_0(t))\) be a solution to \(\dot{x} = f(x, u)\) and consider nearby solution \((x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))\):

\[
\dot{x}(t) = f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t)) \\
= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}(x_0(t), u_0(t))\tilde{x}(t) \\
+ \frac{\partial f}{\partial u}(x_0(t), u_0(t))\tilde{u}(t) + O(\|\tilde{x}, \tilde{u}\|^2)
\]
We hence have for small \((\tilde{x}, \tilde{u})\), approximately

\[
\dot{x}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t)
\]

where (if \(\dim x = 2, \dim u = 1\))

\[
A(t) = \frac{\partial f}{\partial x}(x_0(t), u_0(t)) = \left[ \begin{array}{cc} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{array} \right] \bigg|_{(x_0(t), u_0(t))}
\]

\[
B(t) = \frac{\partial f}{\partial u}(x_0(t), u_0(t)) = \left[ \begin{array}{c} f_1 \\ f_2 \end{array} \right] \bigg|_{(x_0(t), u_0(t))}
\]

Note that \(A(t)\) and \(B(t)\) are time varying! If we linearise around an equilibrium point \((x_0(t), u_0(t)) \equiv (x_0, u_0)\) they become time-invariant \(A\) and \(B\).
Linearisation, continued

Linearisation of output equation

\[ y(t) = h(x(t), u(t)) \]

along the nominal output \( y_0(t) = h(x_0(t), u_0(t)) \) gives

\[ \tilde{y}(t) = C(t)\tilde{x}(t) + D(t)\tilde{u}(t) \]

where \( \tilde{y}(t) = y(t) - y_0(t) \) och (om \( \text{dim } y = \text{dim } x = 2, \text{dim } u = 1 \))

\[ C(t) = \left. \frac{\partial h}{\partial x} \right|_{(x_0, u_0)} = \left[ \begin{array}{c} \frac{\partial h_1}{\partial x_1} \\ \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} \\ \frac{\partial h_2}{\partial x_2} \end{array} \right] \bigg|_{(x_0(t), u_0(t))} \]

\[ D(t) = \left. \frac{\partial h}{\partial u} \right|_{(x_0, u_0)} = \left[ \begin{array}{c} \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_1}{\partial u_2} \\ \frac{\partial h_2}{\partial u_1} \end{array} \right] \bigg|_{(x_0(t), u_0(t))} \]
Example: Rocket

\[ h(t) = v(t) \]
\[ \dot{v}(t) = -g + \frac{v_e u(t)}{m(t)} \]
\[ \dot{m}(t) = -u(t) \]

Let \( u_0(t) \equiv u_0 > 0 \);

\[ x_0(t) = \begin{bmatrix} h_0(t) \\ v_0(t) \\ m_0(t) \end{bmatrix}; \]

\[ m_0(t) = m_0 - u_0 t. \]

Linearisation:

\[ \ddot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -\frac{v_e u_0}{m_0(t)^2} \\ 0 & 0 & 0 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 0 \\ \frac{v_e}{m_0(t)} \\ -1 \end{bmatrix} \tilde{u}(t) \]
The eigenvalues $\lambda(t)$ of $A(t)$ can **not** be used to determine stability:

$$A(t) = \begin{pmatrix}
-1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\
-1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t
\end{pmatrix}, \quad \alpha > 0$$

Eigenvalues are constant

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

with negativ real part for $0 < \alpha < 2$. However, solution to $\dot{x} = A(t)x$ is

$$x(t) = \begin{pmatrix}
e^{(\alpha-1)t} \cos t & e^{-t} \sin t \\
-e^{(\alpha-1)t} \sin t & e^{-t} \cos t
\end{pmatrix} x(0),$$

which is unlimited when $\alpha > 1$. 
Example — Sticksaw

Why can we stabilize an inverted pendulum by just applying vertical oscillations (note, no feedback)?

Watch video www.youtube.com/watch?v=rwGAzyOn0U0
Dynamics for inverted pendulum with sinusoidal movement

\[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \ell^{-1} \left( g + a\omega^2 \sin \omega t \right) \sin x_1
\end{align*}
\]

Periodic trajectory \( x_0(t), u_0(t) \) with period \( T = \frac{2\pi}{\omega} \).

Linearisation along trajectory gives \( \dot{x}(t) = A(t) \dot{x}(t) \) where

\[
A(t) = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ a\omega^2 \sin \omega t \cos x_1 & 0 \end{bmatrix}
\]
There is no analytical solution to $\dot{x} = A(t)x$ (and we can not used the eigenvalues).

Analysis (see course in nonlinear systems) of $\dot{x} = A(t)x$ shows that the system is stable when $\omega$ is sufficiently large.

For the case $a = 1\text{cm}$, $\ell = 17\text{cm}$, stability when $\omega > 182$. 
Simulation gives good agreement with mathematical analysis based on linearisation.

\[ \omega = 183 \]

\[ \omega = 182 \]
Today

- Design Tradeoffs
- Sensitivity Conservation Law
- Linearization around trajectory
  - Example: Stabilization of inverted pendulum

Thanks to Karl Johan Åström