Last week

- Argument principle
- General Nyquist criterion:

$$P^{\text{UNSTABLE}}_{\text{CLOSED}} - P^{\text{UNSTABLE}}_{\text{OPEN}} = \text{CW encirclements of -1}$$

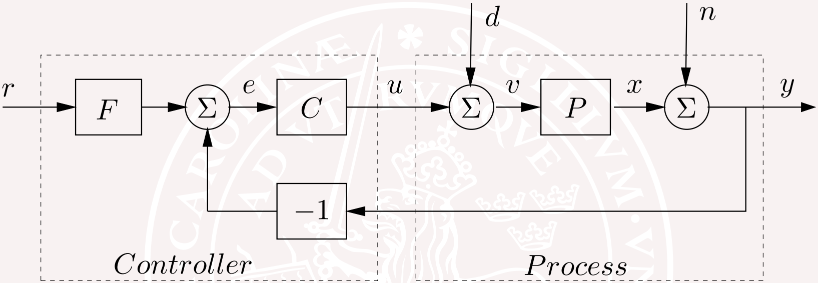
- Bode's relations between gain and phase

Lecture 3

- **Design Tradeoffs**
- Sensitivity Conservation Law
- Linearization around trajectory

  -Example: Stabilization of inverted pendulum

Architecture with Two Degrees of Freedom



Ingredients:

- Controller: feedback C, feedforward F
- Load disturbance d : Drives the system from desired state
- Process: transfer function P
- Measurement noise n : Corrupts information about x
- Process variable x should follow reference r

Typical Requirements

A controller should

A: Reduce effects of load disturbances

B: Not inject too much measurement noise into the system

C: Make the closed loop insensitive to variations in the process

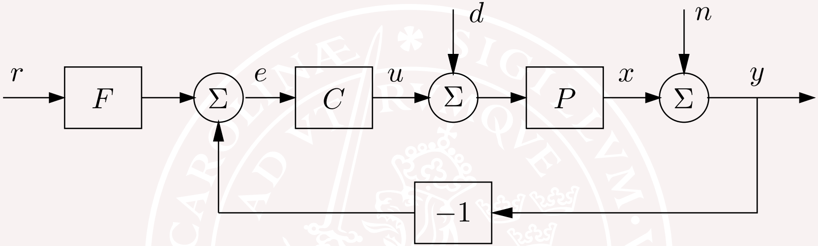
D: Make output follow command signals well

Systems with two degrees of freedom (2DOF)

- Design feedback C for A, B and C
- Then design feed-forward F to handle D

Systems with error feedback ($F = 1$) do not allow this separation of responses to command signal and disturbances.

The Gangs of Four and Seven



$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$E = -\frac{P}{1+PC}D - \frac{1}{1+PC}N + \frac{F}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four GoF)

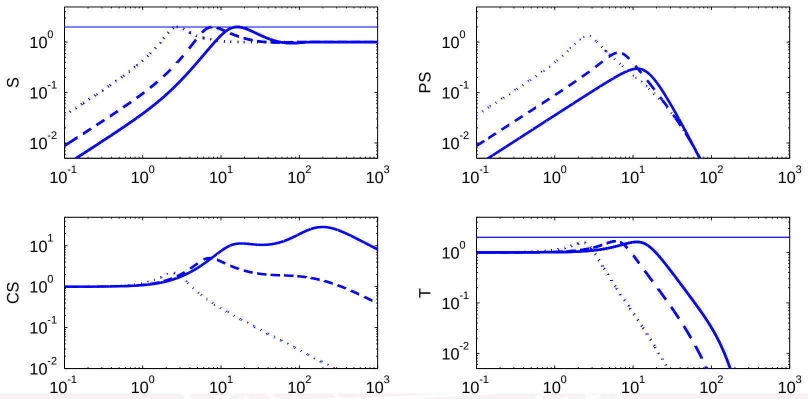
$$\frac{PC}{1+PC} \qquad \frac{P}{1+PC} \qquad \frac{C}{1+PC} \qquad \frac{1}{1+PC}$$

- The system with a controller having two degrees of freedom is characterized by *seven* transfer function (The Gang of Seven GoS)

$$\frac{PCF}{1+PC} \qquad \frac{CF}{1+PC} \qquad \frac{F}{1+PC}$$

- To fully understand a system it is necessary to look at all transfer functions
- It may be strongly misleading to only show properties of a few systems for example the response of the output to command signals, a common omission in literature.

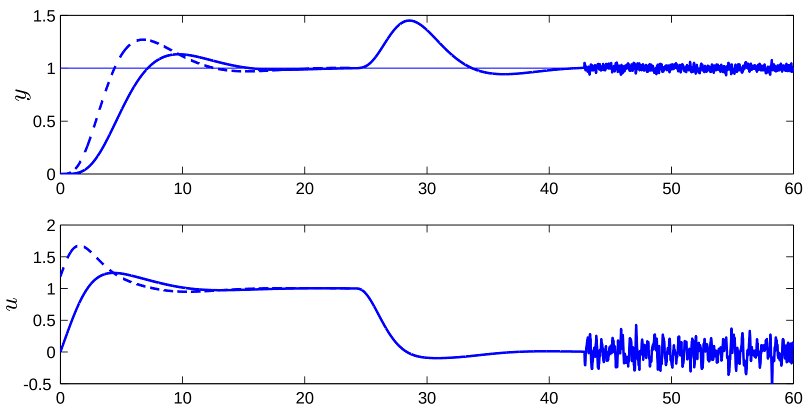
Gain Curves of the Gang of Four



Gain curves of the Gang of Four for a heat conduction process with I (dash-dotted), PI (dashed) and PID (full) controllers.

One plot like this gives a good overview of performance and robustness!

One Way to Show All Responses



Step in r Step in d Measurement noise n

Robustness, small process variations

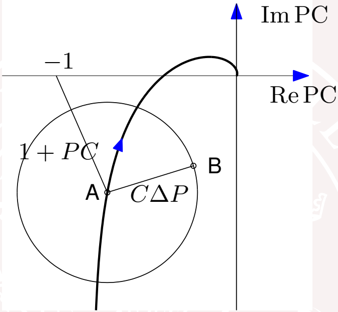
Effect of small process changes on $T = PC/(1+PC)$

$$\frac{dT}{dP} = \frac{C}{(1+PC)^2} = \frac{ST}{P}, \qquad \frac{dT}{T} = S\frac{dP}{P}$$

Robustness: Small relative impact of relative process variations **when S is small**

How much can the process be changed without making the closed loop unstable?

Robustness against large process variations



Closed loop stability with $P(s) + \Delta P(s)$ is guaranteed if nominal loop is stable and

$$|C\Delta P| < |1 + PC| \qquad \Longleftrightarrow \qquad \left|\frac{\Delta P}{P}\right| < \left|\frac{1 + PC}{PC}\right| = \frac{1}{|T|}$$

Robustness: Large variations permitted **when T is small**

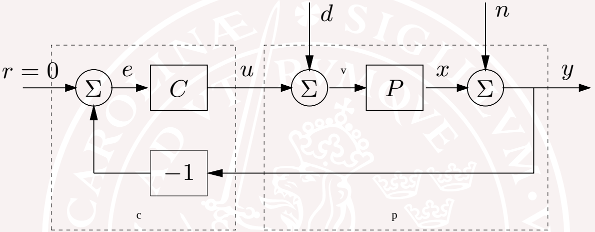
Note $S + T = 1$.

Lecture 3

- Design Tradeoffs
- **Sensitivity Conservation Law**
- Linearization around trajectory

  -Example: Stabilization of inverted pendulum

Effect of Feedback on Disturbances



Output without control: $Y_{ol} = N + PD$

Output with feedback control: $Y_{cl} = \frac{1}{1+PC}(N + PD) = SY_{ol}$

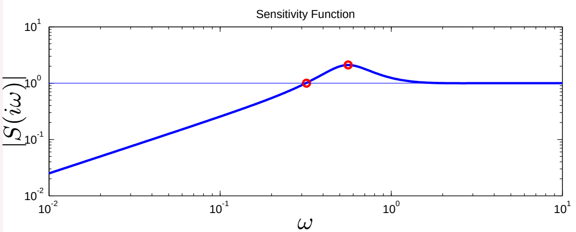
The sensitivity function $S = 1/(1 + PC)$ tells how feedback influences the effect of disturbances: Disturbances with frequencies such that $|S(i\omega)| < 1$ are reduced by feedback, disturbances with frequencies such that $|S(i\omega)| > 1$ are amplified by feedback.

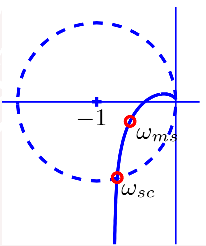
Assessment of Disturbance Reduction

We have

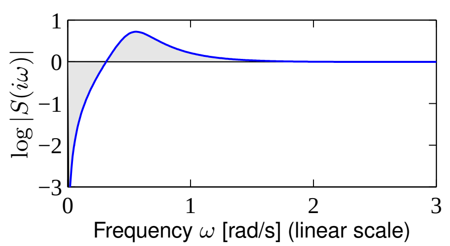
$$Y_{cl} = SY_{ol}(t), \qquad S(s) = \frac{1}{1+P(s)C(s)}$$

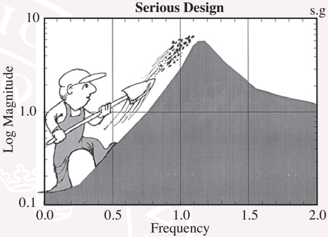
- Feedback attenuates disturbances when $|S(i\omega)| < 1$
- Feedback amplifies disturbances when $|S(i\omega)| > 1$
- The sensitivity crossover frequency ω_{sc} ($|S(i\omega_{sc})| = 1$) is an important parameter, (there may be many values)





The Water Bed Effect - Bode's Integral





If the closed loop is stable and $P(s)C(s)$ has relative degree ≥ 2:

$$\int_0^\infty \log |S(i\omega)| d\omega = \pi \sum_{p_k \in RHP} \operatorname{Re} p_k$$

The sensitivity can be decreased at one frequency at the cost of increasing it at another frequency. Feedback design is a trade-off!.

Lecture 3

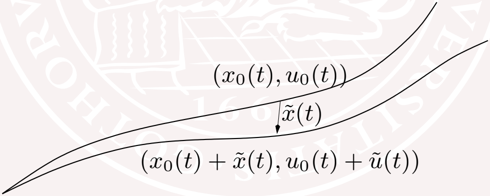
- Design Tradeoffs
- Sensitivity Conservation Law
- **Linearization around trajectory**

  -Example: Stabilization of inverted pendulum

Linearization around a trajectory

Let $(x_0(t), u_0(t))$ be a solution to $\dot{x} = f(x, u)$ and consider nearby solution $(x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$:

$$
\begin{aligned}
\dot{x}(t) &= f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t)) \\
&= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}(x_0(t), u_0(t))\tilde{x}(t) \\
&\quad + \frac{\partial f}{\partial u}(x_0(t), u_0(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)
\end{aligned}
$$



Linearisation, continued

We hence have for small $(\tilde{x}, \tilde{u})$, approximately

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + B(t)\tilde{u}(t)$$

where (if dim $x = 2$, dim $u = 1$)

$$A(t) = \frac{\partial f}{\partial x}(x_0(t), u_0(t)) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \Bigg|_{(x_0(t), u_0(t))}$$

$$B(t) = \frac{\partial f}{\partial u}(x_0(t), u_0(t)) = \begin{bmatrix} \frac{f_1}{u} \\ \frac{f_2}{u} \end{bmatrix} \Bigg|_{(x_0(t), u_0(t))}$$

Note that $A(t)$ and $B(t)$ are **time varying**! If we linearise around an equilibrium point $(x_0(t), u_0(t)) \equiv (x_0, u_0)$ they become time-invariant A and B.

Linearisation, continued

Linearisation of output equation

$$y(t) = h(x(t), u(t))$$

along the nominal outpout $y_0(t) = h(x_0(t), u_0(t))$ gives

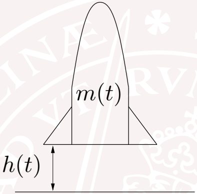
$$\tilde{y}(t) = C(t)\tilde{x}(t) + D(t)\tilde{u}(t)$$

where $\tilde{y}(t) = y(t) - y_0(t)$ och (om dim $y =$ dim $x = 2$, dim $u = 1$)

$$C(t) = \left.\frac{\partial h}{\partial x}\right|_{(x_0,u_0)} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} \end{bmatrix}\right|_{(x_0(t),u_0(t))}$$

$$D(t) = \left.\frac{\partial h}{\partial u}\right|_{(x_0,u_0)} = \left.\begin{bmatrix} \frac{\partial h_1}{\partial u_1} \\ \frac{\partial h_2}{\partial u_1} \end{bmatrix}\right|_{(x_0(t),u_0(t))}$$

Example: Rocket



$$\dot h(t) = v(t)$$
$$\dot v(t) = -g + \frac{v_e u(t)}{m(t)}$$
$$\dot m(t) = -u(t)$$

Let $u_0(t) \equiv u_0 > 0;\;\; x_0(t) = \begin{bmatrix} h_0(t) \\ v_0(t) \\ m_0(t) \end{bmatrix};\; m_0(t) = m_0 - u_0 t$.

Linearisation: $\dot{\tilde x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-v_e u_0}{m_0(t)^2} \\ 0 & 0 & 0 \end{bmatrix} \tilde x(t) + \begin{bmatrix} 0 \\ \frac{v_e}{m_0(t)} \\ -1 \end{bmatrix} \tilde u(t)$

Linear *time varying* systems — Warning!

The eigenvalues $\lambda(t)$ of $A(t)$ can **not** be used to determine stability:

$$A(t) = \begin{pmatrix} -1+\alpha\cos^2 t & 1-\alpha\sin t\cos t \\ -1-\alpha\sin t\cos t & -1+\alpha\sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Eigenvalues are constant

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

with negativ real part for $0 < \alpha < 2$. However, solution to $\dot{x} = A(t)x$ is

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t}\cos t & e^{-t}\sin t \\ -e^{(\alpha-1)t}\sin t & e^{-t}\cos t \end{pmatrix} x(0),$$

which is unlimited when $\alpha > 1$.

Example — Sticksaw

Why can we stabilize an inverted pendulum by just applying vertical oscillations (note, no feedback)?

Watch video `www.youtube.com/watch?v=rwGAzy0noU0`

Stick saw

Dynamics for inverted pendulum with sinusoidal movement

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ell^{-1}\left(g + a\omega^2 \sin \omega t\right) \sin x_1 \end{cases}$$

Periodic trajectory $x_0(t), u_0(t)$ with period $T = 2\pi/\omega$.

Linearisation along trajectory gives $\dot{\tilde{x}}(t) = A(t)\tilde{x}(t)$ where

$$A(t) = \frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ a\omega^2 \sin \omega t \cos x_1 & 0 \end{bmatrix}$$

Stability for pendulum on stick saw

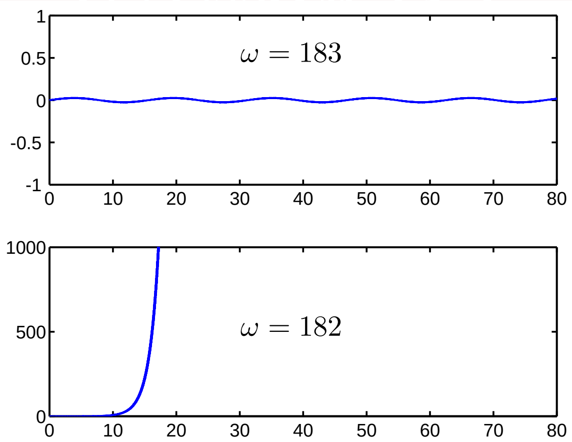
There is no analytical solution to $\dot{x} = A(t)x$ (and we can not used the eigenvalues).

Analysis (see course in nonlinear systems) of $\dot{x} = A(t)x$ shows that the system is stable when ω is sufficiently large.

For the case $a = 1$cm, $\ell = 17$cm, stability when $\omega > 182$

Stick saw —Simulation

Simulation gives good aggreement with mathematical analysis based on linearisation



Today

- Design Tradeoffs
- Sensitivity Conservation Law
- Linearization around trajectory

  -Example: Stabilization of inverted pendulum

Thanks to Karl Johan Åström