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Math background
Laplace transform AK 17
Transient and initial states AK 18
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Presentations HANDIN 1: TBD; no presentations HANDIN2
\[ \int_C f(z)\,dz, \quad C : \{z(t), t \in [a, b]\}, \quad \int_a^b f(z) \frac{dz}{dt} \,dt, \]

**Important example:** \( f(z) = \frac{1}{z-p} \), with \( C : \{z(t) = p + re^{it}, \quad t \in [0, 2\pi]\} \)

\( f(z) \) analytic, closed curve, Cauchy's integral theorem:

different paths same integral, deformation of integration path

\[ \int_C \frac{f(z)}{z-p} \,dz = f(p)2\pi i, \quad \text{Cauchy's integral formula} \]

\( \{p_k\}_{k=1}^n \) poles to \( f(z) \) inside \( C \), then \( \int_C = \int_{C_1} + \cdots + \int_{C_n} \),

Res\( z=p_k f(z) = \frac{1}{2\pi i} \int_{C_k} f(z)\,dz \), residue calculus
Laplace transform

- Double vs single sided Laplace
- Strip of definition. Different for different signals
- Transfer functions. How do we handle different strips of definition?
- Use one sided transforms + analytic continuation
- Makes it possible to also analyse unstable causal systems
Double-sided: Consider time functions $f(t), \ -\infty < t < \infty$

$$F(s) = (\mathcal{L}_{II} f)(s) = \int_{-\infty}^{\infty} e^{-st} f(t) \, dt$$

Converges in strip $\Omega : \alpha < Re \ s < \beta$, $F(s)$ analytic in $\Omega$.

$e^{-\alpha t} f(t) \to 0$, $t \to \infty$, och $e^{-\beta t} f(t) \to 0$, $t \to -\infty$.

Ex $\alpha < 0$ and $\beta > 0$ requires exponential convergence for both $t \to \infty$ and $t \to -\infty$.

Single-sided: Consider $f(t), \ 0 \leq t < \infty$

$$F(s) = (\mathcal{L}_{I} f)(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt$$

Converges in half plane $\Omega : \alpha < Re \ s$, $F(s)$ analytic in $\Omega$.

$e^{-\alpha t} f(t) \to 0$, $t \to \infty$, note $\alpha > 0$ allows $f(t) \to \infty$, $t \to \infty$. 

Bo Bernhardsson  
FRT130 Control Theory, Lecture 1
Laplace transform - example

\[ f(t) = e^{2t}, \ t \geq 0, \quad F = \mathcal{L}_I\{f\}, \quad F(s) = \lim_{T \to \infty} \int_0^T e^{2t} e^{-st} dt \]

\[ F(s) = \lim_{T \to \infty} \left[ \frac{1}{2 - s} e^{(2-s)t} \right]_0^T = \frac{1}{2 - s} \lim_{T \to \infty} \left\{ e^{(2-s)T} - 1 \right\} \]

\[ \lim_{T \to \infty} e^{(2-s)T} = 0, \quad Re \ s > 2 \]

So

\[ F(s) = \frac{1}{s - 2}, \quad Re \ s > 2 \]

Extend domain of definition with analytic continuation to \( \mathbb{C} - \{s = 2\} \), only possible such function is \( F(s) = \frac{1}{s - 2} \)

Nice video about analytic continuation:
www.youtube.com/watch?v=sD0NjbwqlYw&t=3s
Transfer functions for causal systems

Weight function

\[ y(t) = \int_0^t h(t-\tau)u(\tau)d\tau = \int_0^t h(\tau)u(t-\tau)d\tau \]

\[ h(\tau), \quad 0 \leq \tau < \infty \]

\[ G(s) = (\mathcal{L}_I h)(s) \]

\[ Y(s) = G(s)U(s) \]
Laplace transform relations

$$\mathcal{L}_I (f') = sF(s) - f(0)$$

Proof:

$$\mathcal{L}_I \left( \frac{df}{dt} \right) = \int_0^\infty e^{-st} \frac{df}{dt} dt \quad (*)$$

$$= s \int_0^\infty e^{-st} f(t) dt + \left[ e^{-st} f(t) \right]_{t=0}^\infty$$

$$= sF(s) - f(0)$$

(If both integrals converge and if $e^{-st} f(t) \to 0$ as $t \to \infty$).
What is $\mathcal{L}_I (f'')$?

- a $s^2 F(s) - f(0)$
- b $s^2 F(s) - f'(0)$
- c $s^2 F(s) - sf(0) - f'(0)$
- d $s^2 F(s) - sf'(0) - f(0)$
When $s \to 0$ in (*) we get

$$\int_0^\infty \frac{df}{dt} dt = \lim_{s \to 0} sF(s) - f(0)$$

If the limit value $\lim_{t \to \infty} f(t)$ exists, then this can be written

$$\lim_{t \to \infty} f(t) - f(0) = \lim_{s \to 0} sF(s) - f(0)$$

which is the final value theorem
If we instead let $s \to \infty$ we have

$$
\lim_{s \to \infty} \int_{0}^{\infty} e^{-st} \frac{df}{dt} \, dt = \lim_{s \to \infty} sF(s) - f(0)
$$

This motivates that we should have

$$
0 = \lim_{s \to 0^+} sF(s) - f(0)
$$

which is the initial value theorem

Both the final and initial value theorems need conditions to guarantee that the calculations we just did are correct.
If we instead let $s \to \infty$ we have

$$\lim_{s \to \infty} \int_{0}^{\infty} e^{-st} \frac{df}{dt} \, dt = \lim_{s \to \infty} sF(s) - f(0)$$

This motivates that we should have

$$0 = \lim_{s \to 0^+} sF(s) - f(0)$$

which is the initial value theorem.

Both the final and initial value theorems need conditions to guarantee that the calculations we just did are correct.
Initial and Final-value theorems - rational $F$

**Initial Value Theorem** Assume the Laplace transform $F(s)$ is rational and strictly proper. Then

$$
\lim_{t \to +0} f(t) = \lim_{s \to +\infty} sF(s)
$$

**Final Value Theorem.** Assume that $F(s)$ is rational and all poles to $sF(s)$ have negative real part, then

$$
\lim_{t \to +\infty} f(t) = \lim_{s \to +0} sF(s)
$$

Sketch for rational $F(s)$: The theorem is true if $F(s) = (s - p)^k$ (check). Write $F$ as a sum of such terms.
Proof slightly more general final value theorem

\[ \text{On } \frac{f(t)}{e^{at+tk}} \rightarrow C \in \mathbb{R} \text{ as } t \rightarrow \infty \]

\[ g(t) = \frac{f(t)}{e^{at+tk}} \quad \text{begrenzbar für } \forall t \geq 0 \]

\[ F(s) = \int_0^\infty e^{-st} e^{at+tk} g(t) \, dt = \left[ \frac{x=(e^{at+tk})}{s > a} \right] \]

\[ = \int_0^\infty e^{-x} \frac{x^k}{(s-a)^k} g\left(\frac{x}{s-a}\right) \frac{dx}{s-a} \]

\[ \lim_{s \to a^+} (s-a)^k F(s) = \lim_{s \to a^+} \int_0^\infty e^{-x} x^k g\left(\frac{x}{s-a}\right) \, dx \]

\[ = \lim_{t \to \infty} g\left(\frac{t}{s-a}\right) \int_0^\infty e^{-x} x^k \, dx \]

\[ = \lim_{t \to \infty} \frac{f(t)}{e^{at+tk}} \Gamma(k+1) \quad \text{Krone: } k+1 > 0 \]
Transients and initial conditions

\[ \dot{x} = Ax + Bu, \quad x(0) = x_0 \]
\[ y =Cx + Du \]

Laplace transform gives

\[ sX(s) - x_0 = AX(s) + BU(s) \]
\[ X(s) = (sI - A)^{-1}(BU(s) + x_0) \]
\[ Y = [C(sI - A)^{-1}B + D]U(s) + C(sI - A)^{-1}x_0 \]

\[ G(s) \]
Example: Sinusoidal input signal

\[ \dot{x} = -x + u \quad x(0) = x_0 \quad u(t) = \sin t \]

gives after Laplace transform

\[ sX(s) - x(0) = -X(s) + U(s), \quad U(s) = \frac{1}{s^2 + 1} \]

Solving for \( X \) gives

\[ X(s) = \frac{1}{s + 1} (U(s) + x_0) = \frac{1}{s + 1} \left( \frac{1}{s^2 + 1} + x_0 \right) \]

\[ = \frac{0.5 - 0.5s}{s^2 + 1} + \frac{0.5 + x_0}{s + 1} \]

Invers transformation (table) gives

\[ x(t) = \frac{1}{2} \sin t - \frac{1}{2} \cos t + \left( x_0 + \frac{1}{2} \right) e^{-t} \]
Laplace transform in Matlab (or Maple)

```matlab
>> s=tf('s')
>> G = (1-s)/(s^2+s+1)
G =
    -s + 1
    ---------------
    s^2 + s + 1
>> step(G)
```

![Step Response](image)
Laplace transform in Matlab (or Maple)

```matlab
>> clear s
>> syms s t x0

>> ilaplace((1-s)/(s^2+s+1))
ans =
-exp(-t/2)*(cos((3^(1/2)*t)/2) - 3^(1/2)*sin((3^(1/2)*t)/2))

>> ilaplace((0.5-0.5*s)/(s^2+1) + (0.5+x0)/(s+1))
ans =
sin(t)/2 - cos(t)/2 + exp(-t)*(x0 + 1/2)

>> latex(ans)

\frac{\sin(t)}{2} - \frac{\cos(t)}{2} + e^{-t} \left(x0 + \frac{1}{2}\right)
```
A sliding block - where will it stop?

A block is sliding according to

\[ \ddot{y}(t) + cy(t) = 0 \] (1)

with start in position \( y(0) = a \) and speed \( \dot{y}(0) = b \). Determine \( \lim_{t \to \infty} y(t) \).

Laplace transform of (1) gives

\[ s^2 Y(s) - sy(0) - \dot{y}(0) + c[sY(s) - y(0)] = 0 \]

\[ Y(s) = \frac{sy(0) + \dot{y}(0) + cy(0)}{s^2 + cs} \]

Final value theorem gives

\[ \lim_{t \to \infty} y(t) = \lim_{s \to +0} sY(s) = \lim_{s \to +0} \frac{sy(0) + \dot{y}(0) + cy(0)}{s + c} \]

\[ = \frac{\dot{y}(0) + cy(0)}{c} = \frac{b}{c} + a \]

What did we miss? The condition \( c > 0 \).
A block is sliding according to

\[ \ddot{y}(t) + c\dot{y}(t) = 0 \]  

(1)

with start in position \( y(0) = a \) and speed \( \dot{y}(0) = b \). Determine \( \lim_{t \to \infty} y(t) \).

Laplace transform of (1) gives

\[ s^2 Y(s) - sy(0) - \dot{y}(0) + c[sY(s) - y(0)] = 0 \]

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Final value theorem gives

\[ \lim_{t \to \infty} y(t) = \lim_{s \to +0} sY(s) = \lim_{s \to +0} \frac{sy(0) + \dot{y}(0) + cy(0)}{s + c} \]

\[ = \frac{\dot{y}(0) + cy(0)}{c} = \frac{b}{c} + a \]

What did we miss? The condition \( c > 0 \).
Want to solve the differential equation

\[ y^n + a_1 y^{n-1} + \ldots + a_{n-1} y' + a_n y = 0 \]

Characteristic polynomial

\[ a(s) = s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n = 0 \]

If \( a(\alpha) = 0 \) then \( y(t) = C e^{\alpha t} \) is a solution to the differential equation.

The general solution is

\[ y(t) = \sum_k C_k(t) e^{\alpha_k t} \]

where \( C_k(t) \) is a polynomial of degree \( m - 1 \) if \( \alpha_k \) is a root of mult. \( m \).

\( y(t) \to 0 \) if all roots are in the open left half plane.
Eigenvalues - stability

\[ G(s) = C(sI - A)^{-1}B = \frac{1}{\det(sI - A)} C \text{adj}(sI - A)B \]

Eigenvalues: \( \det(sI - A) = 0 \).

\[
\begin{bmatrix}
\dot{\theta}_1 \\
\dot{\theta}_2 \\
\end{bmatrix} = v \begin{bmatrix}
-2 & 0 \\
1 & -1 \\
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
\theta_2 \\
\end{bmatrix} + v \begin{bmatrix}
2 \\
0 \\
\end{bmatrix} u
\]

How do the eigenvalues depend on speed \( v \)?
Frequency analysis

- Frequency curves
  \[ u(t) = \sin \omega t, \quad y(t) = A(\omega) \sin(\omega t + \varphi(\omega)) \]
  \[ A(\omega) = |G(i\omega)|, \quad \varphi(\omega) = \text{arg} \ G(i\omega) \]
- Representation of \( G(s) \) and \( G(i\omega) \)
- Nyquist diagram - complex number \( G(i\omega) \)
- Bode diagram – \( |G(i\omega)| \) and \( \text{arg} \ G(i\omega) \)
  \[ G = G_1 G_2 G_3 G_4 \ldots \]
Course content

Lec1  Basic system theory
Lec2  Argument variation principle, Nyquist theorem, Bode’s relations
Lec3  Stability, Robustness, Sensitivity Function

**w7 Handin 1:** Laplace transform and Frequency plots.

Lec 4  State coordinate change, zeros, state feedback, observers

Lec5  Controllability and Observability, Kalman’s decomposition theorem
Lec6  Linear mappings and least squares problems

**w10: HANLIN 2:** State representations

Presentations HANLIN 1: TBD; no presentations HANLIN2