



**LUND**  
UNIVERSITY

Department of  
**AUTOMATIC CONTROL**

## **Control Theory**

### **Handin 1**

#### **Information About the Hand-in Assignments**

The course FRT130 Control Theory contains two mandatory hand-in assignments. The exercises range from investigations of theoretical concepts to simulation exercises using numerical tools, such as MATLAB.

The intention of the exercises is that you should apply your knowledge from complex and linear analysis in order to solve the tasks given. Some of the exercises have more of a discussion character, and require that you actively use the available literature.

Hand-in Assignment 1 is performed in groups of two or three students. Hand-in Assignment 1 is presented orally at a seminar with other groups. In addition to presenting your solutions, you should also discuss the solutions of other groups. Furthermore, each group is to hand in a written report before the seminar.

#### **Report**

The written report is an important part of the work, which is evaluated both with respect to content and layout. The report should be written such that engineering students on your own level can follow the solutions.

Apply the guidelines for writing reports that you have obtained in previous courses. Some remarks:

- The report can be written with paper and pencil or using appropriate computer software. Write complete sentences.
- If possible, start by describing the problem to be solved. Also, explain the notation and variables that you introduce.
- Explain the different steps of your solution and provide logical motivations for them. Specify the theorems or results that you utilize (Pythagoras Theorem, Cauchy's Integral Formula, ...).
- Please make sure that you verify that all questions in the exercises are answered in your solutions.
- Attach figures and plots if it contributes to the interpretation of the solutions. Also attach relevant code or scripts if you employ numerical tools for simulations and computations.

### **Grading**

In the course FRT130 Control Theory, the grades are Pass and Fail. In order to pass the course, both hand-in assignment reports need to be approved. For the first handin, an acceptable presentation is also necessary.

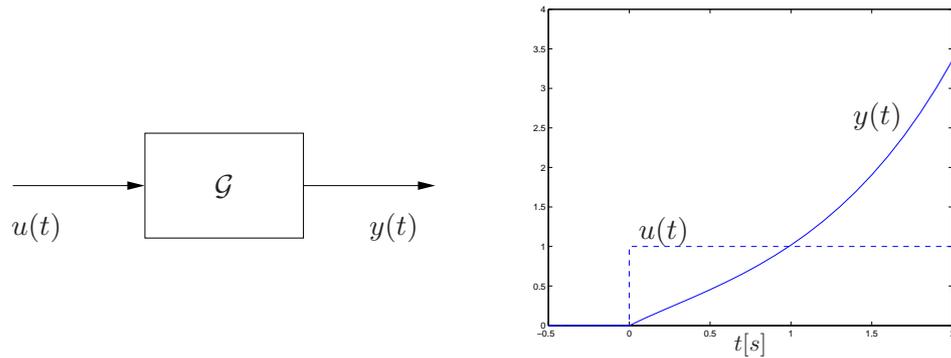


Figure 1: The system and its step response in Exercise 1a).

1.

- a. In an experiment a step function  $u(t) = 1, t \geq 0$ , is applied to a linear time-invariant system  $\mathcal{G}$  at rest ( $x(0) = 0$ ). The step response is shown in Figure 1, which is described by the function

$$y(t) = \frac{1}{2} \left( e^t - \frac{1}{3} e^{-3t} - \frac{2}{3} \right), \quad t \geq 0.$$

Calculate the impulse response.

- b. Compute the transfer function  $G(s)$  for the system  $\mathcal{G}$  in exercise a) by using the definition of the Laplace transform. The Laplace transform  $G(s)$  only converges in a half-plane  $\operatorname{Re} s > \alpha \in \mathbb{R}$ . Determine  $\alpha$ .
- c. Show that the general solution  $y(t)$  for a linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu, \quad x(0) = x_0 \\ y &= Cx + Du \end{aligned}$$

satisfies

$$y(t) = |G(i\omega)| \sin(\omega t + \arg(G(i\omega))) + Ce^{At} (x_0 - \bar{x}_0(\omega)),$$

when  $u(t) = \sin \omega t = (e^{i\omega t} - e^{-i\omega t})/2i, t \geq 0$ , where  $\omega$  is a real number such that  $i\omega$  is not an eigenvalue of  $A$ . In particular, find expressions for the frequency response  $G(i\omega)$  and the part of the transient denoted by  $\bar{x}_0(\omega)$ .

- d. The linearized dynamics for the system with two water tanks which is used in laboration 1 and 2 in the basic course FRT010 Reglerteknik AK can be described by

$$\begin{aligned} \dot{x}_1 &= -\frac{\gamma_1}{2\sqrt{x_1^0}} x_1 + \delta u \\ \dot{x}_2 &= \frac{\gamma_1}{2\sqrt{x_1^0}} x_1 - \frac{\gamma_2}{2\sqrt{x_2^0}} x_2 \end{aligned}$$

where  $x_i$  is the height of tank  $i$ . Choose stationary point ( $x_1^0 = 0.5, x_2^0 = 0.5$ ) and parameter values according to the manual for laboration 2. Simulate the

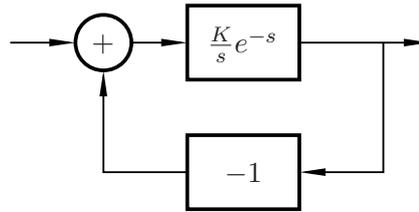


Figure 2: The system in Exercise 2a).

linearized model with the input signal  $u(t) = A \sin(\omega t)$  using MATLAB for various amplitudes  $A$ , frequencies  $\omega$ , and initial conditions  $x_0$ . In particular, investigate what happens when the initial state is chosen so that the transient response vanishes, by using the result of exercise c).

Hint: The function `lsim` in MATLAB can be used to simulate systems on state-space form created by the function `ss`.

2.

- a. An example of a system whose transfer function is not rational is the system in Figure 2. For which  $K > 0$  is this (closed-loop) system asymptotically stable? Sketch the output signal in the time interval  $t \in [0, 3]$  when a step  $r(t) = 1$  is applied to the closed-loop system at  $t = 0$  when the system is at rest.

- b. Consider the open system

$$L(s) = K \frac{(s+2)(s+1.9)}{s^3-1}.$$

Is the system stable? For which  $K > 0$  is the system stabilized by simple negative feedback? Plot the Nyquist curve and use the Nyquist criterion. MATLAB code that might be useful:

```
K=1; L=tf(K*[1,3.9,3.8],[1,0,0,-1]);
nyquist(L)
```

- c. Use the Nyquist criterion to find the values of  $K > 0$  such that the open-loop system

$$L(s) = K \frac{2s+1}{s^2(s+1)(0.4s+1)}$$

is asymptotically stable with simple negative feedback. In particular, discuss the image of the small semi-circle. Code that might be useful:

```
b=[0,0,0,2,1]; a=[0.4,1.4,1,0,0]; L=tf(b,a)
fi=[-1:0.01:1]*pi/2; circ=exp(i*fi);
wmin=0.5;wmax=10;
bs=polyval(b,wmin*circ);
as=polyval(a,wmin*circ);
gcirc=bs./as;
plot(real(gcirc),imag(gcirc))
hold on
nyquist(L,{wmin,wmax})
```

The Nyquist criterion also describes how the *number* of poles in the right half plane varies for different values of  $K > 0$ . Use MATLAB to verify your result by computing the roots of the closed-loop denominator polynomial for interesting values of  $K$ . Code that might be useful:

```
b=[0,0,0,2,1]; a=[0.4,1.4,1,0,0];
roots(a+K*b)
```

What happens if the open-loop system instead is

$$L(s) = K \frac{2s - 1}{s^2(s + 1)(0.4s + 1)}?$$

**3.**

- a. Formulate Bode's relations and describe the important mathematical tools used in their proofs.
- b. A version of Bode's relations that is useful for numerically computing the phase given the gain is

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi} \int_0^\infty \frac{\log |G(i\omega)| - \log |G(i\omega_0)|}{\omega^2 - \omega_0^2} d\omega,$$

where  $\log$  refers to the natural logarithm. Use this formula to compute the phase of the transfer function

$$G_1(s) = \frac{1}{s + 1}.$$

In MATLAB numerical integration can be done using the function `integral`.

What happens for

$$G_2(s) = \frac{1}{s - 1}?$$

Compare your results with the corresponding values obtained with the analytical expression for the phase.

- c. Consider the Bode diagram for

$$G(s) = \frac{(s + 1/2)(s + 2)}{s(s + 1/4)(s + 1)(s + 4)}.$$

Find a  $\hat{G}(s)$  whose gain  $|\hat{G}(i\omega)|$  is a straight line in a loglog diagram and which approximates  $|G(i\omega)|$  between  $\omega = 1/4$  and  $\omega = 4$ . That is, find  $K$  and  $a$  in

$$\hat{G}(s) = \frac{K}{s^a},$$

where the slope  $a$  does not have to be an integer. Compute the phase  $\arg G(i)$  and compare with  $\arg \hat{G}(i)$ . Discuss the connection between the slope of the line and the phase. Code that might be useful:

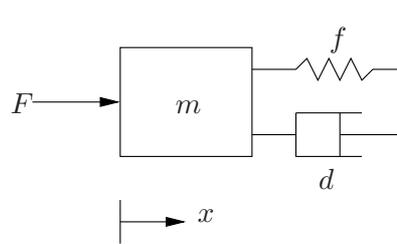


Figure 3: The system in Exercise 4.

```

om=logspace(log10(1/4),log10(4));
F_mag=K./om.^a;
loglog(om,F_mag)

```

4. The mechanical system in Figure 3 with a mass, spring, and damper is described by

$$m\ddot{x} = -d\dot{x} - fx + F$$

- a. Find the transfer function from  $F$  to  $x$ .
- b. Let the system be controlled by a proportional controller with gain  $K$ . Compute the sensitivity function  $S(s)$  when  $m = f = d = 1$ . What are the different interpretations of  $S(s)$ ?
- c. Use MATLAB to plot the Bode diagram of  $S(s)$  for the three cases  $K = 1, 5, 10$ . Discuss the relation of the plots to Bode's formula.