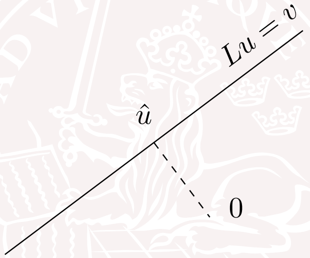
Lecture 6

- Least squares problem, under-determined
- Measures of controllability
- Least squares problem, over-determined
- Measures of observability
- Example: Function approximation

Least squares problems I

Given linear operator L and vector v, minimize $|u|$ under the constraint $Lu = v$.



The operator L is “short and fat”: More variables than equations.

Many solutions, want the shortest one.

Notice the right angle in the picture

Linear operators and Hilbert spaces

If L is a matrix and u and v are vectors in a finite-dimensional space this is an easy matrix problem: The solution is

$$\hat{u} \;=\; L^T(LL^T)^{-1}v$$

We want to generalize to a situation were we optimize over e.g. "all possible control signals $u([0,T])$".

Linear operators and Hilbert spaces

Useful theory: Linear operators in infinite-dimensional vector spaces, scalar product $\langle x, y\rangle$, "orthogonal" means that $\langle x, y\rangle = 0$.

This theory is very useful, not only in control and signal processing.

Dont have time to present the mathematical background and detail, only some intuition and the resulting formulas for the optimal solution.

For more detail, see Lecture 6 in the PhD course Linear System theory
`www.control.lth.se/Education/DoctorateProgram/linear-systems.html`

Linear operators and adjoints

Given a (continuous) linear operator L from a Hilbert space to another, the adjoint L^* is an operator defined by the relation

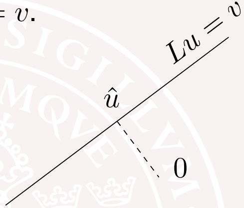
$$\langle Lu, v \rangle = \langle u, L^* v \rangle$$

for all u, v.

This generalizes the matrix transpose in the finite dimensional case

Least squares problem I

Minimize $|u|$ under the constraint$Lu = v$.



Solution: $\hat{u}$ must satisfy $L\hat{u} = v$ and

$$0 = \langle \hat{u}, \hat{u} - u \rangle \text{ for all } u \text{ with } Lu = v$$

If LL^* is invertible then the (in this case unique) solution can be written

$$\hat{u} \quad = \quad L^*(LL^*)^{-1}v$$

Application: Reach wanted state $x(T)$ with minimal control signal

Measure of controllability

$$Lu = \int_0^{t_1} e^{A(t_1-t)} Bu(t)dt$$

$$L^*x(t) = \left[e^{A(t_1-t)}B\right]^T x \quad \text{(easy to check that } \langle x, Lu\rangle = \langle L^*x, u\rangle)$$

$$W := LL^* = \int_0^{t_1} e^{A(t_1-t)} BB^T e^{A^T(t_1-t)} dt = \int_0^{t_1} e^{A\tau} BB^T e^{A^T\tau} d\tau$$

The problem of controlling the system $\dot{x} = Ax + Bu$ from $x(0) = 0$ to $x(t_1) = x_1$ with minimal cost $\|u\|^2 = \int_0^{t_1} u^2 dt$ hence has the solution

$$\hat{u}(t) = L^*(LL^*)^{-1}x_1 = B^T e^{A^T(t_1-t)} W^{-1} x_1$$

and the minimal squared cost $\|\hat{u}\|^2$ equals

$$x_1^T (LL^*)^{-1} x_1 = x_1^T W^{-1} x_1.$$

Controllability Gramian

The matrix

$$W = \int_0^{t_1} e^{A\tau} B B^T e^{A^T\tau} d\tau$$

is called Gramian. The cost of reaching the state x_1 is $x_1^T W^{-1} x_1$. The smallest eigenvalue of W is a measure of controllability, since $1/\lambda_{\min}(W)$ is the control signal (squared) norm that is needed to reach all states having norm one.

For the case $t_1 = \infty$ and A stable, one can calculate W from the Lyapunov equation (`W=lyap(A,B*B')` in matlab)

$$WA^T + AW + BB^T = 0.$$

Example: Gramian for trailer

$$A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e^{At} = \begin{bmatrix} e^{-t} & 0 \\ te^{-t} & e^{-t} \end{bmatrix}$$

$$\begin{aligned} W &= \int_0^{t_1} \begin{bmatrix} e^{-t} \\ te^{-t} \end{bmatrix} \begin{bmatrix} e^{-t} \\ te^{-t} \end{bmatrix}^T dt \\ &= \frac{1}{4} \begin{bmatrix} 2 - 2e^{-2t_1} & 1 - (2t_1 + 1)e^{-2t_1} \\ 1 - (2t_1 + 1)e^{-2t_1} & 1 - (2t_1^2 + 2t_1 + 1)e^{-2t_1} \end{bmatrix} \end{aligned}$$

For $t_1 = \infty$ we get

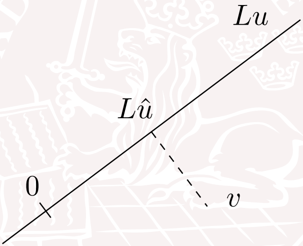
$$W = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$$

with eigenvalues 0.65 and 0.096.

Least squares problem II

Given L and v, minimize $|Lu - v|$ with respect to u.

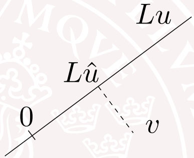
L is "tall and thin": More equations than variables



Notice the right angle !

Least squares problems II

Minimize $|Lu - v|$ with respect to u.



Solution: $\hat{u}$ must satisfy

$$0 = \langle Lx, L\hat{u} - v\rangle \text{ for all } x$$

Equivalently

$$L^*L\hat{u} \;=\; L^*v$$

Observability

$$\begin{cases} \dfrac{dx}{dt} = Ax, \quad x(0) = x_0 \\ \quad y = Cx \end{cases}$$

The system is observable if x_0 uniquely can be determined from $y_{[0,t_1]}$.

$$y(t) = Ce^{At}x_0 = (Mx_0)(t), \quad y = Mx_0$$
$$M : \mathbf{R}^n \to \mathbf{L}_2^p[0, t_1]$$

The operator M, maps x_0 to y, i.e. from an n-dimensional space to a space of functions

Measure of observability

If $y = Mx_0 + e$, i.e. if true value disturbed by measurement noise e, then the equations can typically not be solved exactly.
Least squares solution:

$$\min_{x_0} ||y - Mx_0||$$

$$W = M^* M = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$$

$$\hat{x}_0 = (M^* M)^{-1} M^* y = W^{-1} \int_0^{t_1} e^{A^T (t_1 - t)} C^T y(t) dt$$

If $M\hat{x}_0 = Mx_0 + e$ then the estimation error $\tilde{x}_0 = x_0 - \hat{x}_0$ satisfies

$$\tilde{x}_0^T M^* M \tilde{x}_0 = \|e\|^2$$

The smallest eigenvalue to the *observability gramian* $W = M^* M$ gives a measure of observability. If it is close to zero, then small e can give large $\tilde{x}_0$ (bad).

Other example: Function approximation

Choose the real numbers u_0, u_1, u_2 to minimize $\int_0^1 |e^t - u_0 - u_1 t - u_2 t^2|^2 dt$

Solution:

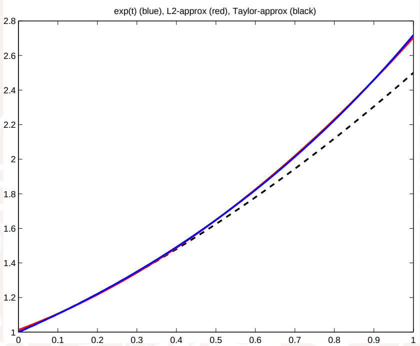
$$u = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}, \qquad Lu = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ u_2 \end{bmatrix}, \qquad v(t) = e^t$$

$$L^* v = \int_0^1 \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} e^t dt = \begin{bmatrix} e-1 \\ 1 \\ e-2 \end{bmatrix}$$

$$L^* L = \int_0^1 \begin{bmatrix} 1 \\ t \\ t^2 \end{bmatrix} \begin{bmatrix} 1 & t & t^2 \end{bmatrix} dt = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

$$\widehat{u} = (L^* L)^{-1} L^* v = \begin{bmatrix} 1.013 \\ 0.851 \\ 0.839 \end{bmatrix}$$

Example: Funcion approximation



Notice that the least squares approximation (red)

$$e^t \approx 1.013 + 0.851t + 0.839t^2$$

is much better than the Taylor approximation (black)

$$e^t \approx 1 + t + 0.5t^2$$

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