Last week

- Design Tradeoffs
- Sensitivity Conservation Law
- Linearization around trajectory

  -Example: Stabilization of inverted pendulum

Lecture 4

Use more linear algebra to understand state space realisations

- State Space Realizations (pp 139-150)
- $G(s)$, denominator and numerator, poles and zeros
- Change of coordinates, diagonal and controllable form
- State-feedback
- Observers
- Feedback from estimated states
- Integral action by disturbance model

State-space realization

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$sX(s) - x_0 = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$[sI - A]X(s) = x_0 + BU(s),$$
$$Y(s) = \underbrace{\left(C[sI - A]^{-1}B + D\right)}_{G(s)} U(s) + \left(C[sI - A]^{-1}\right)x_0$$

$$y(t) = (g * u)(t) + Ce^{At}x_0$$

Char. polynomial: $\det[sI - A] = s^n + a_1 s^{n-1} + \ldots + a_n$

Poles and Zeros

Pole λ, eigenvalue to A, eigenvector v

$$\frac{dx}{dt} = Ax + Bu, \qquad Av = v\lambda,\ e^{At}v = ve^{\lambda t}$$

$$u = 0, x(0) = v \Rightarrow x(t) = e^{At}x(0) = ve^{\lambda t}$$

Zero z: Exists x_0, so that $x(0) = x_0$ och $u(t) = u_0 e^{zt}$ gives $x(t) = x_0 e^{zt}$, and $y(t) = y_0 e^{zt} = 0$.

$$zx_0 e^{zt} = Ax_0 e^{zt} + Bu_0 e^{zt} \Leftrightarrow [A - zI]x_0 + Bu_0 = 0$$

$$y(t) = Cx(t) + Du(t) = (Cx_0 + Du_0)e^{zt} = 0$$

$$\underbrace{\begin{pmatrix} A - zI & B \\ C & D \end{pmatrix}}_{M(z)} \begin{pmatrix} x_0 \\ u_0 \end{pmatrix} = 0$$

z is called a zero if the matrix $M(z)$ looses rank.

Matlab: `eig([A,B;C,D],diag([1,1,0]))` (for $n = 2$)

Diagonalisation by coordinate change

If we have n linear independent eigenvectors then

$$A\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix} = \begin{pmatrix} v_1\lambda_1 & \dots & v_n\lambda_n \end{pmatrix} = \underbrace{\begin{pmatrix} v_1 & \dots & v_n \end{pmatrix}}_{=:T^{-1}}\Lambda$$

$$TAT^{-1} = \Lambda$$

$$Te^{At}T^{-1} = e^{\Lambda t}$$

If $\dot{x} = Ax$, the coordinate change $z = Tx$ gives $\dot{z} = \Lambda z$.

$$G(s) = C(sI - A)^{-1}B = \underbrace{CT^{-1}}_{\overline{C}}(sI - \underbrace{TAT^{-1}}_{\overline{A}})^{-1}\underbrace{TB}_{\overline{B}}$$

Coordinate change to diagonal form

Using the coordinate change T we get the diagonal form (parallell systems)

$$\dot{z} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} z + \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{pmatrix} u$$

$$y = \begin{pmatrix} \gamma_1 & \gamma_2 & \dots & \gamma_n \end{pmatrix} z$$

$$G(s) = \frac{\beta_1\gamma_1}{s-\lambda_1} + \ldots + \frac{\beta_n\gamma_n}{s-\lambda_n}$$

Corresponds to partial fraction decomposition of $G(s)$

Controllable form

A system with this structure is said to be on controllable form

$$\dot{x} = \begin{pmatrix} -a_1 & -a_2 & \dots & -a_{n-1} & -a_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u$$

We see that

$$\begin{pmatrix} s + a_1 & a_2 & \dots & a_{n-1} & a_n \end{pmatrix} X(s) = U(s)$$

$$sX_{k+1}(s) = X_k(s) \Rightarrow X_{n-k}(s) = s^k X_n(s), \quad k = 1, \dots, n-1$$

$$(s^n + a_1 s^{n-1} + \dots + a_{n-1}s + a_n)X_n(s) = U(s)$$

Controllable form, cont'd

Since

$$
\begin{aligned}
y &= \begin{pmatrix} b_1 & b_2 & \dots & b_{n-1} & b_n \end{pmatrix} x \\
Y(s) &= b_1 X_1(s) + b_2 X_2(s) + \cdots + b_n X_n(s) \\
&= (b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n) X_n(s)
\end{aligned}
$$

we get

$$
Y(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_{n-1} s + a_n} U(s)
$$

State Feedback

$$\begin{cases} \frac{dx}{dt} = Ax + Bu \\ y = Cx \end{cases}$$

$$u = -Lx + l_r r$$

$$\begin{cases} \frac{dx}{dt} = (A - BL)x + Bl_r r \\ y = Cx \end{cases}$$

State Feedback on Controllable Form

$$\frac{dx}{dt} = \begin{pmatrix} -a_1 - \tilde{l}_1 & -a_2 - \tilde{l}_2 & \dots & -a_{n-1} - \tilde{l}_{n-1} & -a_n - \tilde{l}_n \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & & 1 & 0 \end{pmatrix} x + \begin{pmatrix} l_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} r$$

$$y = \begin{pmatrix} b_1 & b_2 & \cdots & b_n \end{pmatrix} x$$

$$p_i = a_i + \tilde{l}_i$$

$$G(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + \ldots + b_n}{s^n + p_1 s^{n-1} + \ldots + p_n} l_r$$

Can change denominator (poles) with L. No change of numerators (zeros).

Zero and state feedback

Zeros are never changed with state feedback

$$\begin{pmatrix} A - BL - zI & Bl_r \\ C & 0 \end{pmatrix} = \begin{pmatrix} A - zI & B \\ C & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -L & l_r I \end{pmatrix}$$

Looses rank for the same z. Hence same zeros.

Observers

$$\frac{dx}{dt} = Ax + Bu$$

$$y = Cx + n$$

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

$$\tilde{x} = x - \hat{x}$$

$$\frac{d\tilde{x}}{dt} = (A - KC)\tilde{x} - Kn$$

K: tradeoff between estimator convergence speed and noise gain

"Duality" between state feedback and observers

$$A \leftrightarrow A^T$$

$$B \leftrightarrow C^T$$

$$L \leftrightarrow K^T$$

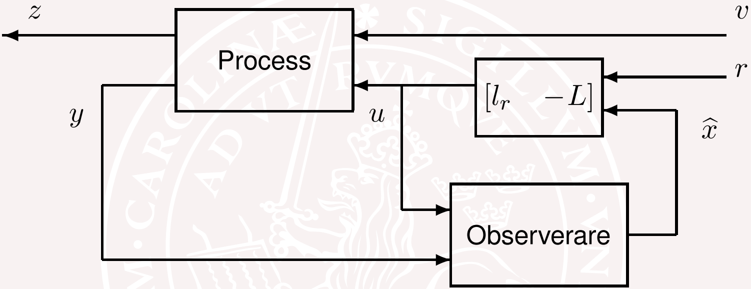
Observer form

$$\frac{dz}{dt} = \begin{pmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{pmatrix} z + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \end{pmatrix} z$$

Gives $A - KC$ a nice structure.

Feedback from estimated states



Process: $\begin{cases} \dot{x}(t) = Ax(t) + B[u(t) + v(t)] \\ y(t) = Cx(t) + n(t) \end{cases}$

Observer: $\begin{cases} \frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) + l_r r(t) \end{cases}$

Closed loop system properties

Eliminate u and y:

$$\frac{d}{dt}x(t) = Ax(t) - BL\widehat{x}(t) + B(l_r r(t) + v(t))$$
$$\frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) - BL\widehat{x}(t) + K[Cx(t) - C\widehat{x}(t)] + Kn(t)$$

Introduce $\widetilde{x} = x - \widehat{x}$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \widetilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(k) \\ \widetilde{x}(k) \end{bmatrix} + \begin{bmatrix} B(l_r r(t) + v(t)) \\ -Kn(t) \end{bmatrix}$$

Two kind of poles to the closed loop system:

Process poles: $0 = \det(sI - A + BL)$

Observer poles: $0 = \det(sI - A + KC)$

Example – Double integrator

$$\frac{dx}{dt} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Choose for example

$$\det(sI - A + BL) = \det \begin{pmatrix} s & -1 \\ l_1 & s + l_2 \end{pmatrix} = s^2 + s + 1$$

i.e. $l_1 = 1, l_2 = 1$ and

$$\det(sI - A + KC) = \det \begin{pmatrix} s + k_1 & -1 \\ k_2 & s \end{pmatrix} = s^2 + 2s + 4$$

i.e. $k_1 = 2, k_2 = 4$.

Example – Double integrator cont'd

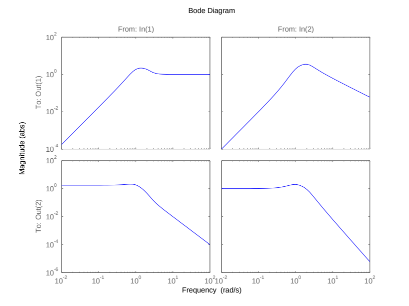
Process $P(s) = \frac{1}{s^2}$ with feedback

$$C(s) = L(sI - A + BL + KC)^{-1}K$$

gives closed loop system transfer functions

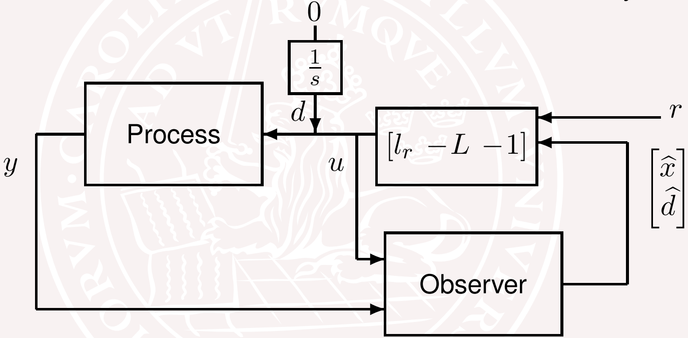
$$\begin{bmatrix} \frac{1}{1+PC} & \frac{P}{1+PC} \\ \frac{C}{1+PC} & \frac{PC}{1+PC} \end{bmatrix}$$

with Bode's amplitude diagrams (Gang of Four)



Integral action - alternative method

Instead of the method in AK (lec9) one can extend the system with a disturbance model and let the Kalman filter estimate a stationary offset



$$u = l_r r - L\widehat{x} - \widehat{d}$$

Integral action - design method

Introduce extended state space vector $x_e := \begin{bmatrix} x \\ d \end{bmatrix}$.

Design Kalman filter for the extended system

$$\dot{x}_e = \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} x_e + \begin{bmatrix} B \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} C & 0 \end{bmatrix} x_e$$

and use control signal

$$u = l_r r - L\widehat{x} - \widehat{d}$$

It can be shown that this gives integral action in the resulting controller.