LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Exam in Systems Engineering/Process Control

2017-06-02

Points and grading

All answers must include a clear motivation. Answers may be given in English or Swedish. The total number of points is 20 for Systems Engineering and 25 for Process Control. The maximum number of points is specified for each subproblem. Preliminary grading scales:

Systems Engineering:		Process control:	
Grade 3:	10 points	Grade 3:	12 points
4:	14 points	4:	17 points
5:	17 points	5:	21 points

Accepted aid

Authorized *Formelsamling i reglerteknik / Collection of Formulae*. Standard mathematical tables like TEFYMA. Pocket calculator.

Results

The solutions will be posted on the course home page, and the results will be transferred to LADOK. Date and location for display of the corrected exams will be posted on the course home page.

1. Figure 1 shows step responses of six systems. Match them to the following six transfer functions. (3 p)

$$G1 = \frac{10}{s+10}e^{-s} \qquad G2 = \frac{4}{s^2+1.2s+4}$$

$$G3 = \frac{1}{s(s+1)} \qquad G4 = \frac{4}{s^2+2.4s+4}$$

$$G5 = \frac{4}{s^2-3.6s+4} \qquad G6 = \frac{1}{s+1}$$

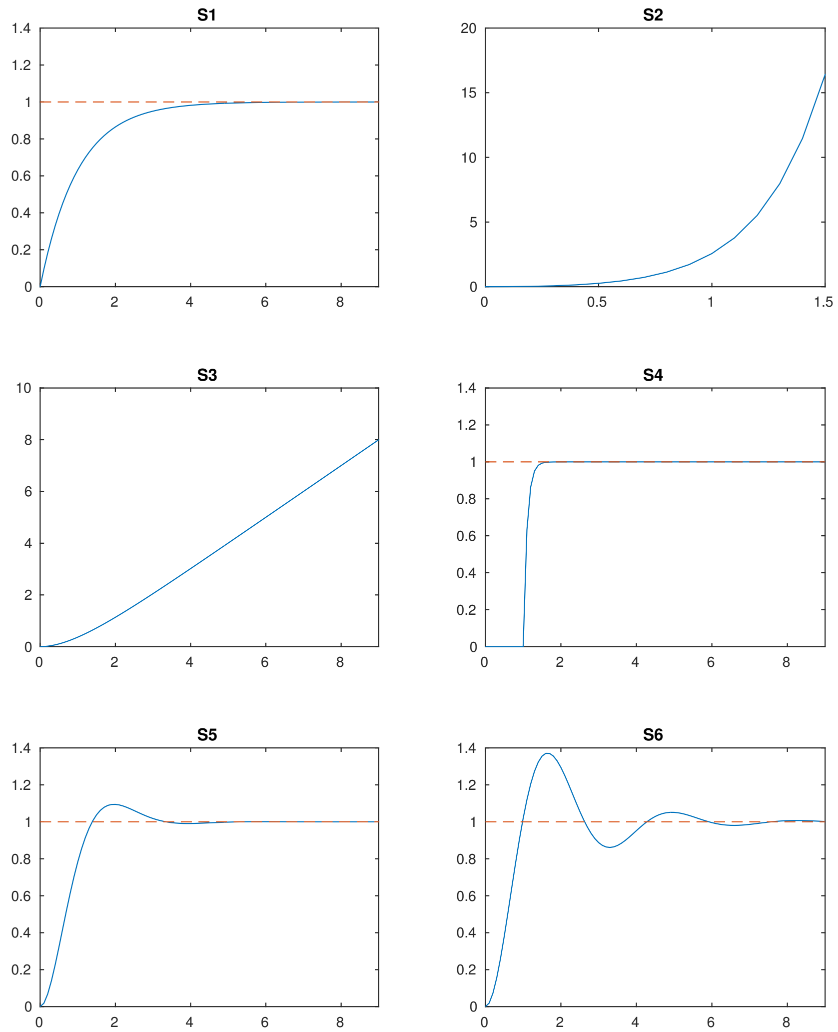

Figure 1 Step responses for the six transfer functions in Problem 1

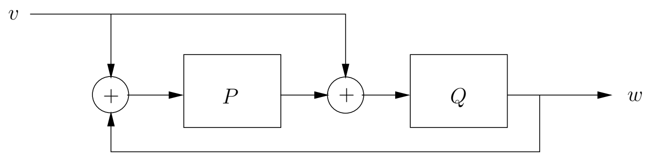

Figure 2 Block diagram for Problem 2 with two block components.

Solution

1. S1 = G6. first order system with static gain 1.
2. S2 = G5. Unstable system.
3. S3 = G3. Integrator
4. S4 = G1. First order system with a time delay.
5. S5 = G4. Second order system with high damping.
6. S6 = G2. Second order system with low damping.

2. Compute the transfer function from v to w for the block diagram in Figure 2.

(2 p)

Solution

Breaking the loop at B gives

$$\begin{aligned} B &= Q(P(B+A)+A) = QPB + Q(P+1)A \\ &\Leftrightarrow B(1-QP) = Q(P+1)A \\ &\Leftrightarrow B = \frac{Q(P+1)}{1-QP}A \end{aligned}$$

The transfer function from A to B is hence

$$\frac{Q(P+1)}{1-QP}$$

3. Figure 3 depicts some characteristics of three different second order systems, all of the form

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} e^{-sL}$$

but with different parameters ζ, ω_0, and L. Combine each Nyquist plot (i)-(iii), with a bode diagram (A)-(C), a step response (1)-(3), and a singularity diagram (I)-(III) in Figure 3 with clear motivations.

(4 p)

Solution

The correct combinations are

- $(i) \leftrightarrow (C) \leftrightarrow (2) \leftrightarrow (I)$
- $(ii) \leftrightarrow (B) \leftrightarrow (3) \leftrightarrow (II)$
- $(iii) \leftrightarrow (A) \leftrightarrow (1) \leftrightarrow (III)$

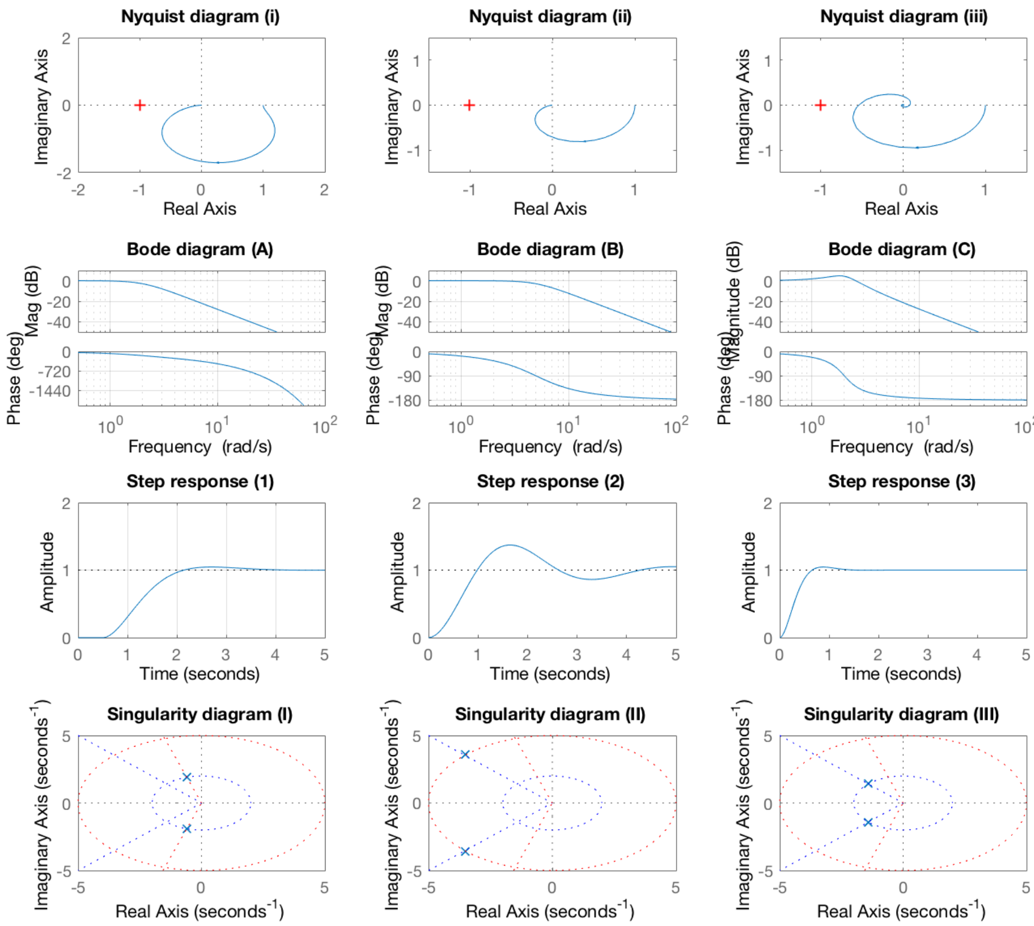

Figure 3 Characteristics of the three second order transfer functions in Problem 3.

4. Consider the system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 \\ -1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

a. Determine the transfer function from u to y. (1.5 p)

b. Determine the poles of the system. Is the system unstable, marginally stable or asymptotically stable? (1.5 p)

Solution

a. The transfer function from u to y is given by $G(s) = C(sI - A)^{-1}B$.

$$G(s) = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} s+1 & 0 \\ 0 & s+2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{(s+1)(s+2)} = \frac{1}{s^2+3s+2}$$

b. The poles of the system are the eigenvalues of the system matrix A, which are also the zeros of the characteristic function. Solving $(s+1)(s+2)=0$ shows that the system has one pole in -1 and one pole in -2. The system is asymptotically stable since the real parts of all its poles are strictly negative.

5. Consider a system with an input signal $u(t)$ and a measurement signal $y(t)$. Let

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{(s+1)^2}$$

be the transfer function from the input $U(s) = \mathscr{L}\{u(t)\}$ to the output $Y(s) = \mathscr{L}\{y(t)\}$. Using the final value theorem,

a. Compute the output $y(t)$ as $t \to \infty$ if the input is a unit step. (1 p)

b. Compute the output $y(t)$ as $t \to \infty$ if the input is a unit ramp. (1 p)

c. Compute the difference $d(t) = u(t) - y(t)$ as $t \to \infty$ if the input is a unit ramp. (1 p)

Solution

According to the final value theorem,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{1}$$

if $sY(s)$ is asymptotically stable.

a. As the Laplace transform of a step $u(t) = \theta(t)$ is $U(s) = s^{-1}$, consequently

$$sY(s) = sG(s)U(s) = s\frac{1}{(s+1)^2}\frac{1}{s} = \frac{1}{(s+1)^2} \tag{2}$$

We have two poles in -1, is therefore asymptotically stable and we compute

$$y(\infty) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} \frac{1}{(s+1)^2} = 1 \tag{3}$$

b. If instead the input is a unit ramp, the $\mathscr{L}$-transform of $u(t) = t\theta(t)$ is $U(s) = s^{-2}$,

$$sY(s) = sG(s)U(s) = s\frac{1}{(s+1)^2}\frac{1}{s^2} = \frac{1}{s(s+1)^2} \tag{4}$$

We now have two poles in -1 an additional pole in $s=0$, the system is marginally stable due to the extra integrator and the final value theorem does not apply.

c. The difference may be written $D(s) = U(s) - Y(s)$ in the Laplace domain, and we now investigate the stability of

$$sD(s) = s(U(s)-Y(s)) = s(1-G(s))U(s) = \left(1 - \frac{1}{(s+1)^2}\right)\frac{1}{s} = \frac{s^2+2s}{(s+1)^2 s} = \frac{s+2}{(s+1)^2} \tag{5}$$

Due to the pole-zero cancellation, we again have two poles in -1 and asymptotic stability. Again applying the final value theorem

$$d(\infty) = \lim_{s\to 0} sD(s) = \lim_{s\to 0} \frac{s+2}{(s+1)^2} = 2 \tag{6}$$

While we in b) could not say anything about the final value of $y(t)$ when using a ramp as an input, it is now clear that difference between the input ramp and response will be a constant 2 as $t \to \infty$.

6. The ideal pendulum of length l $[m]$ in Figure 4 is governed by the nonlinear dynamical state equations

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -(g/l)\sin(x_1)$$

where x_1 is the angle, x_2 is the angular velocity, and g $[m/s^2]$ denotes the constant of gravity.

a. Find *all* stationary points, $\mathbf{x}^0 = [x_1^0, x_2^0]^T$, of the system. (1 p)

b. Give a physical explanation of the location of the stationary points. (1 p)

c. Linearize the system around an arbitrary stationary point $\mathbf{x}^0 = [x_1^0, x_2^0]^T$ by introducing the variables $\Delta x_1 = x_1 - x_1^0$, $\Delta x_2 = x_2 - x_2^0$. (1 p)

d. Decide for each linearized system if it is stable, marginally stable, or unstable. Give a physical explanation of the result. (2 p)

Solution

a. Consider the dynamics on the form $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, with $\mathbf{x} = [x_1, x_2]^T$. In stationarity, $\dot{\mathbf{x}} = \mathbf{0}$, and solving $\mathbf{f}(\mathbf{x}) = 0$,

$$\begin{cases} 0 = x_2 \\ 0 = -(g/l)\sin(x_1) \end{cases} \Leftrightarrow \begin{cases} x_1 = \arcsin(0) = n\pi, \quad n \in \mathbb{Z} \\ x_2 = 0 \end{cases}$$

the equilibrium points of the system are found in $[x_1^0, x_2^0]^T = [n\pi, 0]^T, \quad n \in \mathbb{Z}$.

b. The angular velocity x_2 must be zero in stationarity. The (angular) position of the pendulum x_1 is either upward/inverted (odd n) or downward (even n).

c. The linearized dynamics are given by a Taylor expansion of $\mathbf{f}$ around the linearization point, resulting in the LTI system

$$\dot{\Delta\mathbf{x}} = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}}\bigg|_{\mathbf{x}^0} \Delta\mathbf{x} = \begin{bmatrix} 0 & 1 \\ -(g/l)\cos(x_1^0) & 0 \end{bmatrix} \Delta\mathbf{x} \triangleq \mathbf{A}\Delta\mathbf{x}$$

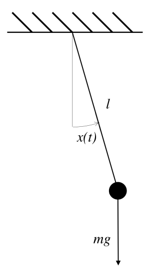

Figure 4 Pendulum in Problem 6.

d. The eigenvalues of the linearized system are given by the roots of the characteristic equation

$$\det(\mathbf{I}\lambda - \mathbf{A}) = \lambda^2 + (g/l)\cos(x_1^0) = 0 \Leftrightarrow \lambda_{1,2} = \pm\sqrt{-\frac{g}{l}\cos(x_1^0)}$$

Plugging in the linearization point, this may equivalently be written

$$\lambda_{1,2} = \pm\sqrt{-\frac{g}{l}\cos(n\pi)} = \pm\sqrt{-\frac{g}{l}(-1)^n}$$

If n is even, the expression under the square root is negative, and the eigenvalues are on the imaginary axis and the system is marginally stable. If n is odd, the expression under the square root is positive, and the eigenvalues are on the real axis on either side of the origin. Therefore, the system is unstable.

This is reasonable since in the inverted position (odd n), the pendulum is likely to be repelled from the stationary points. In the downward position (even n), the pendulum oscillates back and forth.

7. **Only for Process Control:** Consider the multivariable system

$$G(s) = \begin{pmatrix} \frac{2e^{-s}}{s+4} & \frac{e^{-s}}{(s+1)(s+5)} \\ \frac{1}{2s+5} & \frac{3}{s+5} \end{pmatrix}$$

The system should be controlled using two PID controllers.

a. Calculate the relative gain array, RGA, for the system. (1 p)

b. Determine how should the inputs and outputs be paired? Comment on the interaction. (1 p)

c. Find a decoupling matrix that decouples the system dynamics in stationarity and gives the corresponding decoupled system static gains of one. Is the decoupler realizable? (1 p)

Solution

a. The stationary gain matrix is given by

$$G(0) = \begin{bmatrix} 0.5 & 0.2 \\ 0.2 & 0.6 \end{bmatrix}.$$

From this we can calculate the RGA as

$$\text{RGA} = G(0). * (G(0)^{-1})^T = \begin{bmatrix} 1.15 & -0.15 \\ -0.15 & 1.15 \end{bmatrix}.$$

b. The inputs and outputs should be paired so that the corresponding relative gains are positive and as close to one as possible. Hence, we should control the first output using the first control signal and the second output using the second control signal. The coupling is weak, since the RGA is close to identity matrix.

c. That the system is decoupled in stationarity and has static gains of one if D is obtained from the following

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = G(0)D.$$

From this we see that the static decouples should be the inverse of $G(0)$.

$$D = \begin{bmatrix} 2.31 & -0.77 \\ -0.77 & 1.92 \end{bmatrix}.$$

Since the decoupler is static it is obviously realizable.

8. **Only for Process Control:** Consider a first order system

$$\dot{y}(t) = -ay(t) + bu(t)$$

with $a > 0$ and $b > 0$.

a. Discretize the system using *forward Euler* with sampling time h. State the resulting difference equation. (0.5 p)

b. Find the range of $h > 0$, possibly as a function of a and b, such that the discretized system in **a.** is stable. (0.5 p)

c. Discretize the system using *backward Euler* with sampling time h. State the resulting difference equation. (0.5 p)

d. Find the range of $h > 0$, possibly as a function of a and b, such that the discretized system in **c.** is stable. (0.5 p)

Solution

a. Forward Euler gives

$$\frac{y(kh+h) - y(kh)}{h} = -ay(kh) + bu(kh).$$

Rearranging gives

$$y(kh+h) = (1 - ah)y(kh) + bhu(kh).$$

b. The difference equation is stable if $|1 - ah| < 1$, which implies

$$0 < h < 2/a.$$

c. Backward Euler gives

$$\frac{y(kh) - y(kh-h)}{h} = -ay(kh) + bu(kh).$$

Rearranging gives

$$y(kh) = \frac{1}{1+ah} y(kh-h) + bhu(kh).$$

d. The difference equation is stable if $|\frac{1}{1+ah}| < 1$, which implies $h > 0$.