Solutions to exam in Systems Engineering and Process Control 2016-05-31

1.

$G_1 - E$	First order system, time constant 5s, no time delay.
$G_2 - N/A$	First order system, time constant 50s.
$G_3 - D$	Second order system, fully damped, time delay, static gain 1.
$G_4 - A$	Same as G_3 but with static gain 5.
$G_5 - F$	Unstable.
$G_6 - C$	Second order system, oscillative, static gain 1.
$G_7 - N/A$	The same as G_6 but static gain 5.
$G_8 - B$	First order system, time constant 0.2s, time delay 5s, static gain 1.
$G_9 - N/A$	Second order system,fully damped, no time delay.

2.

a. Introducing the states $x_1 = y$ and $x_2 = \dot{y}$ the system equations can be written as

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\gamma/\alpha & -\beta/\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \delta/\alpha \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

b. Laplace transforming the expression gives

$$\alpha s^2 Y + \beta s Y + \gamma Y = \delta U \Rightarrow$$
$$Y = \frac{\delta}{\alpha s^2 + \beta s + \gamma} \quad U$$

c. To get an asymptotically stable system we need all coefficients of the characteristic polynomial to be of the same sign. Rewriting the system to

$$Y = \frac{\frac{\delta}{\alpha}}{s^2 + \frac{\beta}{\alpha} s + \frac{\gamma}{\alpha}} \quad U$$

it is easily seen that the restrictions are that $\beta/\alpha > 0$ and $\gamma/\alpha > 0$.

3.

a. The static gain is $P(0) \approx 2$.

b. The amplitude margin is ≈ 2.

c. To determine the steady state error $e(\infty)$ for a reference change, we need to know how e depends on r. The block diagram gives us:

$$E(s) = \frac{1}{1 + KP(s)} R(s).$$

Therefore, the stationary error $e(\infty)$ is given by

$$e(\infty) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + KP(s)} \frac{1}{s} = \frac{1}{1 + KP(0)} = \frac{1}{1 + 2K}.$$

d. A stationary error $|e(\infty)|$ less than 0.1 then gives

$$\left|\frac{1}{1+2K}\right| < 0.1$$

with solutions $K > 4.5$ and $k < -5.5$. But as the closed loop system becomes unstable for $K > 2$ (the amplitude margin is ≈ 2), the requirement on stability and the stationary error is impossible to fulfill for $K > 0$.

4.

a. The stationary points are those that have zero derivative, i.e., the points that satisfy

$$\begin{aligned} 0 &= x_1(1-x_1)+x_2 & &(= f_1(x_1,x_2)) \\ 0 &= x_2(2-x_1) & &(= f_2(x_1,x_2)) \end{aligned}$$

From the second equation, we get that $x_2 = 0$ or $x_1 = 2$. If $x_2 = 0$, the first equation gives that $x_1 = 0$ or $x_1 = 1$. If $x_1 = 2$, the first equation gives that $x_2 = 2$. Therefore, the stationary points are $(0,0),(1,0),(2,2)$.

b. The only stationary point with $x_0^1 \neq 0$ and $x_0^2 \neq 0$ is the point $(2,2)$. So, we linearize around this. The partial derivatives are

$$\begin{aligned} \frac{\partial f_1}{\partial x_1} &= 1-2x_1, & \frac{\partial f_1}{\partial x_2} &= 1 \\ \frac{\partial f_2}{\partial x_1} &= -x_2, & \frac{\partial f_2}{\partial x_2} &= 2-x_1. \end{aligned}$$

Introduce the new variables

$$\begin{aligned} \Delta x_1 &= x_1 - x_1^0 \\ \Delta x_2 &= x_2 - x_2^0. \end{aligned}$$

where $x_1^0 = 2$ and $x_2^0 = 2$, and let $\Delta x = [\Delta x_1 \ \Delta x_2]^T$. The linearized system becomes

$$\dot{\Delta x} = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \Delta x.$$

c. Let A be the linearized dynamics matrix from subproblem **b.**, i.e.,

$$A = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}.$$

The eigenvalues of the dynamics matrix A are given by the λ that satisfy

$$\det(\lambda I - A) = 0.$$

That is

$$(\lambda+3)\lambda + 2 = \lambda^2 + 3\lambda + 2 = 0.$$

Since all coefficients are positive, the eigenvalues of the A matrix are in the left half-plane, and the linearized system is asymptotically stable.

5.

a. The output signal y is given by

$$y = Dd + PFd + PC(r - y - n) \Rightarrow$$
$$y = \frac{1}{1+PC}(PCr + (D+PF)d - PCn).$$

This gives that

$$G_{r\to y} = \frac{PC}{1+PC}$$
$$G_{d\to y} = \frac{D+PF}{1+PC}$$
$$G_{n\to y} = \frac{-PC}{1+PC}$$

b. No it is not possible, since the transfer functions from the reference signal and from the noise are the same (only differs in the sign). So, if you would want to remove the effect of the noise (i.e. have $G_{n\to y} = 0$) the transfer function from the reference to the output would also be zero. (You could, however, design a controller that has perfect following of references in steady state $G_{r\to y}(0) = 1$ that has a very low gain for higher frequencies and hence attenuates noise of these frequencies well.)

c. The impact from d will disappear if $G_{d\to y} = 0$, i.e., if $D + PF = 0$. Hence, we can design our feedforward controller as

$$F(s) = -\frac{D(s)}{P(s)} = \frac{s+1}{s^2+3s+4}.$$

6.

a. The zero is directly identified as -2. The poles are roots to the denominator, i.e., s that solve $s^2 + 2s + 5 = 0$:

$$s = -1 \pm \sqrt{1-5} = -1 \pm 2i.$$

Since all poles are strictly in the left half-plane, the process is asymptotically stable.

b. The open loop transfer function $G_o(s) = KG_P(s)$ is

$$G_o(s) = \frac{K(s+2)}{s^2+2s+5} = \frac{Q}{P}$$

The characteristic equation is given by

$$Q + P = s^2 + (2+K)s + 2K + 5$$

The desired characteristic equation is

$$(s+3)(s+7) = s^2 + 10s + 21$$

Identifying coefficients leads to the equation system

$$10 = 2 + K$$
$$21 = 2K + 5$$

with solution $K = 8$, that holds for both equations. It is thus possible to place the poles at -3 and -7 using a proportional controller of gain $K = 8$.

c. No, a P-controller is not sufficient since it is not possible to place the poles wherever you want to. The pole placement leads to an equation system with two equations and one variable K. This is in general not solvable.

7.

a.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y4 \end{pmatrix} = \begin{pmatrix} P_1\alpha & 0 \\ 0 & P_2\beta \\ P_3P_1\alpha & P_3(1-\beta) \\ P_4(1-\alpha) & P_4P_2\beta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

b.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_1\alpha & 0 \\ 0 & P_2\beta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

As can be seen this subsystem is decoupled and you should pair u_1 to y_1 and u_2 to y_2.

c. This subsystem is

$$\begin{pmatrix} y_3 \\ y4 \end{pmatrix} = \begin{pmatrix} P_3P_1\alpha & P_3(1-\beta) \\ P_4(1-\alpha) & P_4P_2\beta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

The transfer function matrix in steady state with the values given will become

$$G(0) = \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix}.$$

This gives the RGA matrix

$$\begin{aligned} \Lambda = G(0). * (G(0))^{-T} &= \begin{pmatrix} 1/4 & 3/4 \\ 3/4 & 1/4 \end{pmatrix} . * \frac{1}{1/16 - 9/16} \begin{pmatrix} 1/4 & -3/4 \\ -3/4 & 1/4 \end{pmatrix} \\ &= -2 \begin{pmatrix} 1/16 & -9/16 \\ -9/16 & 1/16 \end{pmatrix} = \begin{pmatrix} -1/8 & 9/8 \\ 9/8 & -1/8 \end{pmatrix} \end{aligned}.$$

From this RGA matrix it is seen that you should pair y_3 with u_2 and y_4 with u_1 since you want elements close to 1.

8.

a. The closed loop system is written as

$$Y(s) = \frac{25}{s+25} R(s)$$

or rearranged as

$$(s+25)Y(s) = 25R(s).$$

Inverse Laplace transform then yields

$$\frac{dy(t)}{dt} + 25y(t) = 25r(t)$$

and a forward approximation of the derivative with sample period $h = 0.1$ then gives

$$\frac{y(0.1k+0.1)-y(0.1k)}{0.1} + 25y(0.1k) = 25r(0.1k)$$

or arranged as a difference equation

$$y(0.1k+0.1) = -1.5y(0.1k) + 2.5r(0.1k)$$

b. The discretization in a is unstable since the coefficient in front of the previous y-value has an absolute value greater than 1. The continuous time system is stable. Hence, the step-responses do not agree. This can be solved by reducing the sampling time or by using another approximation method instead, such as the backward difference.