Lösningar till tentamen i Systemteknik/Processreglering 2014-08-28

1. In stationarity, all time derivatives are zero. This gives:

$$
\begin{aligned}
0 &= 0.5V_0 + 0.1R_0 - 0.1V_0^2 \Rightarrow R_0 = V_0^2 - 5V_0 \\
0 &= 0.5R_0 - 4V_0 - \frac{2}{300}R_0V_0 \\
\Rightarrow 0 &= 0.5V_0^2 - 2.5V_0 - 4V - \frac{2}{300}V_0^3 + \frac{10}{300}V_0^2 \\
\Rightarrow 0 &= V_0(V_0^2 - 5V_0 - 75V_0 + 975) \\
\Rightarrow (V,R) &= (0,0),(15,150),(65,3900)
\end{aligned}
$$

For linearization of the system, the new variables ΔV and ΔR are introduced:

$$
\begin{aligned}
\Delta V &= V - V_0 \\
\Delta R &= R - R_0
\end{aligned}
$$

The expressions for $\dot{V}$ and $\dot{R}$ are linearized with respect to V and R which gives the following expression for the linearized system:

$$
\begin{pmatrix} \dot{\Delta V} \\ \dot{\Delta R} \end{pmatrix} = \begin{pmatrix} 0.5 - 0.2V_0 & 0.1 \\ -4 - \frac{2}{300}R_0 & 0.5 - \frac{2}{300}V_0 \end{pmatrix} \begin{pmatrix} \Delta V \\ \Delta R \end{pmatrix}
$$

Values for the relevant stationary point are inserted:

$$
\begin{pmatrix} \dot{\Delta V} \\ \dot{\Delta R} \end{pmatrix} = \begin{pmatrix} -12.5 & 0.1 \\ -30 & \frac{2}{30} \end{pmatrix} \begin{pmatrix} \Delta V \\ \Delta R \end{pmatrix}
$$

The system's characteristic polynomial is calculated to determine its stability properties:

$$
P(s) = (s + 12.5)(s - \frac{2}{30}) + 3 = s^2 + \frac{373}{30}s + \frac{65}{30}
$$

As all terms in the characteristic polynomial (in a 2nd order system) are positive, the system is asymptotically stable.

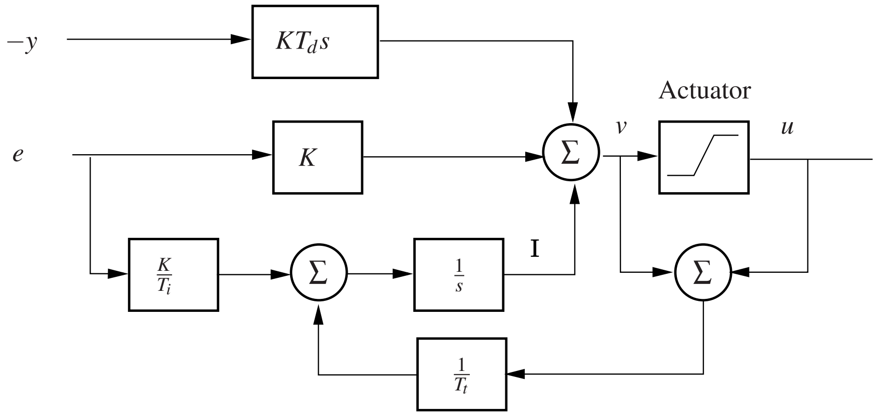
2. The characteristic polynomial is $s^3 + 3s^2 + 4s + 10$. Since $3 > 0$, $4 > 0$, $10 > 0$ and $3 \cdot 4>10$, all poles are strictly in the left half-plane, according to Routh–Hurwitz stability criterion for a third-order polynomial.

3. The poles of $G_1(s)$ are $s = \pm i$, which means that the system is stable but not asymptotically stable. The only step response that could correspond to this system is F.

The second system $G_2(2)$ is asymptotically stable and its static gain is one, which means that the system must correspond to either A, C or E. Since it is a first order system it cannot have any overshot, which rules out C and E. The relative degree of the system is one, which means that the initial derivative och the step response is nonzero. Either of the above mentioned motivations gives that $G_2(s)$ must correspond to A.

The third system is also asymptotically stable. The static gain is zero, which means it must corresponds to step response D.

The fourth and final system is a asymptotically stable second-order system with a time-delay of one time unit. The static gain is one and the relative damping is $\zeta = 0.5$, which means that the step response will have an overshot and will settle at the amplitude 1. The only system that has an overshot and a time-delay is E.



Figur 1 PID controller with anti-windup.

4.

$$G_{cl}(s) = \frac{KG(s)}{1+KG(s)} = \frac{K}{(s-2)(s+4)+K} = \frac{K}{s^2+2s-8+K}$$

For stability, the terms a and b in the characteristic polynomial on the form s^2+as+b must be non-negative, i.e. K must be at least 8.

5 a. The controller is a PI controller. A limitation in the actuator is causing the control signal to saturate, which will cause wind-up of the integral part in the controller. At time $t \in [0,20]$ the controller fails to match the output with the reference, this indicates either that a P controller with too low gain is implemented or that actuator saturation has occurred in a controller with an integral part. At time $t \in [20,30]$, the control signal stays high for ten seconds after the decrease in reference, indicating that the problem is integrator wind-up.

b. Integrator wind-up can be mitigated with an anti wind-up scheme. The amount of actuator saturation is fed back to the input of the integrator which will stop the integrator from obtaining a higher value, see Figure 1.

6 a. The Laplace transform of the differential equation is

$$sY+Y-U=0 \Rightarrow Y(s) = \frac{1}{s+1}U(s) = G_P(s)U(s)$$

The process has a single pole in -1 which indicates that the process is asymptotically stable.

b. The Laplace transform of the PI controller is

$$U(s) = K\left(1+\frac{1}{sT_i}\right)E(s) = \frac{sKT_i+K}{sT_i}E(s) = G_R(s)E(s)$$

The closed-loop transfer function from $R(s)$ to $Y(s)$ is

$$\begin{aligned} Y(s) &= \frac{G_P(s)G_R(s)}{1+G_P(s)G_R(s)}R(s) \\ &= \frac{sKT_i+K}{sT_i(s+1)+sKT_i+K} \\ &= \frac{sK+K/T_i}{s^2+s(K+1)+K/T_i} \end{aligned}$$

The pole specification $-2 \pm i$ yields a characteristic polynomial $(s+2+i)(s+2-i) = s^2+4s+5$, which in turn determines $K = 3$ and $T_i = \frac{3}{5}$

c. The response of a linear system to a sinusoidal input $A\sin(\omega t)$ is

$$|G(i\omega)|A\sin(\omega t + arg(G(i\omega)))$$

The frequency function, its magnitude and argument are

$$\begin{aligned}
G(i\omega) &= \frac{5+3i\omega}{(2+(1+\omega)i)(2+(1-\omega)i)} \\
|G(i\omega)| &= \frac{\sqrt{25+9\omega^2}}{\sqrt{4+(\omega+1)^2}\sqrt{4+(\omega-1)^2}} \\
arg(G(i\omega)) &= \arctan\frac{3\omega}{5} - \arctan\frac{\omega+1}{2} - \arctan\frac{\omega-1}{2}
\end{aligned}$$

Evaluated at the frequency $\omega = 1$

$$\begin{aligned}
|G(i\omega)| &= \sqrt{\frac{17}{16}} \approx 1.03 \\
arg(G(i\omega)) &\approx -0.245 \\
y(t) &= |G(i\omega)|2\sin(\omega t + arg(G(i\omega))) \\
&= 2.06\sin(t - 0.245)
\end{aligned}$$

7. As shown in figure 2, the phase is -180 degrees at $\omega \approx 13$. There the gain is about 0.4, implying that the gain margin is approximately 2.5. The gain is 1 at $\omega \approx 2.6$. There the phase is about -65 degrees, implying that the phase margin is about 115 degrees.

8 a. The stationary gain matrix is given by

$$G(0) = \begin{bmatrix} a & 1 \\ 3 & 2 \end{bmatrix}. \tag{1}$$

From this we can calculate the RGA as

$$\text{RGA} = G(0).*(G(0)^{-1})^T = \frac{1}{2a-3}\begin{bmatrix} 2a & -3 \\ -3 & 2a \end{bmatrix}. \tag{2}$$

b. For $a = 0$ we have

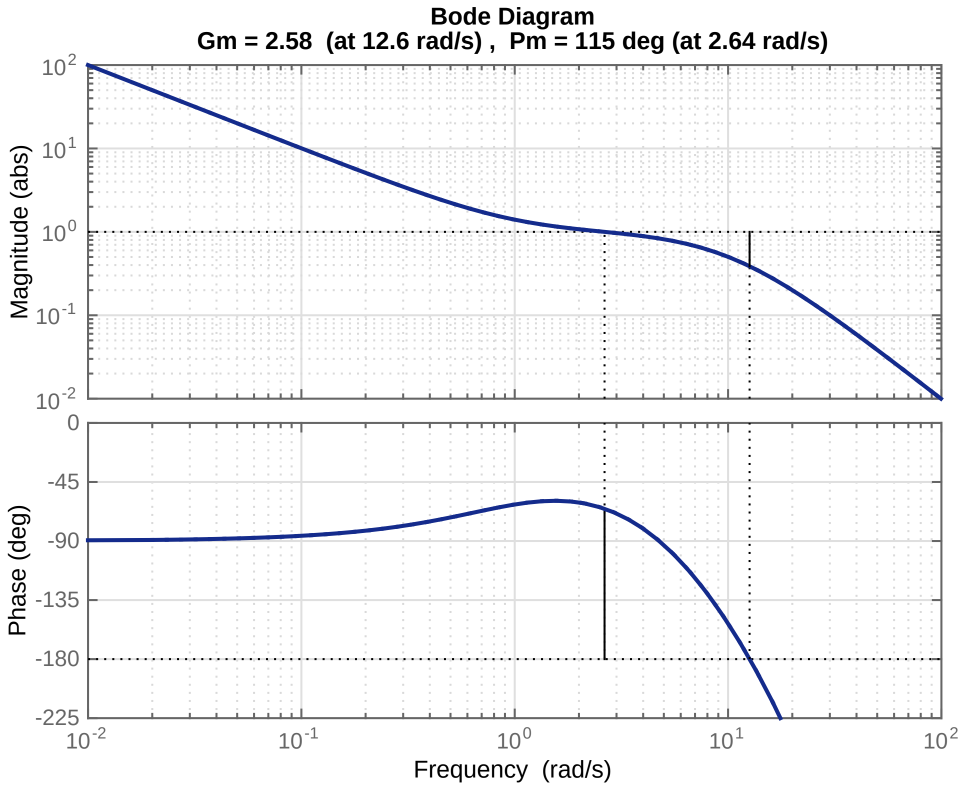
$$\text{RGA} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{3}$$

and from this we see that we should control the first output using the second control signal and the second output using the first control signal. The system is decoupled in stationarity.

c. For $a = 3$ we have

$$\text{RGA} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}. \tag{4}$$

The inputs and outputs should be paired so that the corresponding relative gains are positive and as close to one as possible. Hence, we should control the first output using the first control signal and the second output using the second control signal. The system has difficult interaction.



Figur 2 Gain and phase margins.

d. That the system is decoupled in stationarity and has static gains of one is equivalent to

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = G(0)D. \tag{5}$$

From this we see that the static decouples should be the inverse of $G(0)$.

$$D = \begin{bmatrix} 2/3 & -1/3 \\ -1 & 1 \end{bmatrix}. \tag{6}$$

Since the decoupler is static it is obviously realizable.

9 a. Forward approximation gives

$$\frac{y(kh+h)-y(kh)}{h} + 2y(kh) = 3u(kh) \Leftrightarrow$$
$$y(kh+h) = (1-2h)y(kh) + 3hu(kh)$$

b. Backward approximation gives

$$\frac{y(kh)-y(kh-h)}{h} + 2y(kh) = 3u(kh) \Leftrightarrow$$
$$y(kh) = \frac{1}{1+2h}(y(kh-h) + 3hu(kh))$$