Lösningar till tentamen i Systemteknik 2014-05-27

1 a. Laplace transformation of the differential equation gives:

$$s^2Y(s)+3sY(s)+5Y(s)+2sU(s)=8U(s) \leftrightarrow Y(s)=\frac{-2s+8}{s^2+3s+5}U(s)$$

The characteristic polynomial is s^2+3s+5. Since $3>0$ and $5>0$, the system is asymptotically stable.

b. The system has two poles. It is of second order.

c. The differential equation is a linear combination of $y(t)$ and $u(t)$ and time derivatives of these. Hence, the system is linear.

2 a. The zero is given by

$$8s-3=0 \leftrightarrow s=\frac{3}{8}.$$

The poles are given by

$$s^2+4s-1=0 \leftrightarrow s=-2\pm\sqrt{4+1} \leftrightarrow s_1=-2+\sqrt{5}, s_2=-2-\sqrt{5}.$$

Since one pole is in the right half-plane, the process is not stable.

b. We have the following relation between the input and the output:

$$Y(s)=G_p(S)U(s) \leftrightarrow s^2Y(s)+4sY(s)-Y(s)=8sU(s)-3U(s)$$

Inverse Laplace transformation gives the differential equation

$$\ddot{y}+4\dot{y}-y=8\dot{u}-3u.$$

3. Introduce $f_1(x_1,x_2,u)$ and $f_2(x_1,x_2,u)$ such that

$$\begin{aligned}\dot{x}_1 &= x_1x_2-x_2^2u && = f_1(x_1,x_2,u)\\ \dot{x}_2 &= -(x_1-2)^2+\sqrt{x_1}\sqrt{x_2} && = f_2(x_1,x_2,u).\end{aligned}$$

a. The stationary points can be found by solving

$$f_1(x_1^0,x_2^0,u^0)=0, \quad f_2(x_1^0,x_2^0,u^0)=0.$$

Inserting $u^0=1$ and starting with the first equation we get

$$x_2^0\left(x_1^0-x_2^0\right)=0$$

from which we see that $x_2^0=0$ or $x_2^0=x_1^0$. Evaluation of the second equation with $x_2^0=0$ gives us

$$\left(x_1^0-2\right)^2=0$$

from which we see that $x_1^0=2$. Evaluation of the second equation with $x_2^0=x_1^0$ gives us

$$-\left(x_1^0-1\right)^2+x_1^0=0 \Leftrightarrow \left(x_1^0\right)^2-5x_1^0+4=0.$$

Solving the equation we obtain $x_1^0=1$ or $x_1^0=4$.

The system has three stationary points: $(2,0)$ $(1,1)$ and $(4,4)$.

b. The system should be linearized around $x_1^0 = x_2^0 = 4$ and $u^0 = 1$. The partial derivatives needed are

$$\frac{\partial f_1}{\partial x_1} = x_2, \qquad \frac{\partial f_1}{\partial x_2} = x_1 - 2x_2 u \qquad \frac{\partial f_1}{\partial u} = -x_2^2$$
$$\frac{\partial f_2}{\partial x_1} = -2(x_1 - 2) + \frac{\sqrt{x_2}}{2\sqrt{x_1}}, \qquad \frac{\partial f_2}{\partial x_2} = \frac{\sqrt{x_1}}{2\sqrt{x_2}} \qquad \frac{\partial f_2}{\partial u} = 0$$

Introduce the new variables $\Delta x_1 = x_1 - x_1^0$, $\Delta x_2 = x_2 - x_2^0$, $\Delta u = u - u^0$. Evaluation of the partial derivatives in the stationary point gives the following state-space representation of the linearized system:

$$\dot{\Delta x} = A\Delta x + B\Delta u$$

where

$$A = \begin{bmatrix} 4 & -4 \\ -3.5 & 0.5 \end{bmatrix}, \qquad B = \begin{bmatrix} -16 \\ 0 \end{bmatrix}$$

c. The poles are given by the eigenvalues of the A-matrix.

$$\det(sI - A) = \begin{vmatrix} s-4 & 4 \\ 3.5 & s-0.5 \end{vmatrix} = s^2 - 4.5s - 12 = 0$$

By solving this equation we obtain the poles:

$$s_1 = -1.88, \quad s_2 = 6.38$$

One pole lies in the right half-plane and the linearized system is therefore unstable.

4. The input has the angular frequency $\omega = 3$ rad/s. From the Bode plot, we read the approximate values $|G(3i)| = 0.1$ and $\arg(G(3i)) = -33^\circ = -0.58$ rad. Hence, we get the output
$y(t) = 0.3\sin(3t - 0.58)$

5 a. Since both the controller and valve are open-loop stable the series connection will also be stable. According to the Nyquist criterium the closed-loop system is then stable, since the Nyquist curve does not encircle -1.

b. The magnitude of the system for the frequency ω_0 where the Nyquist curve intersects the real axis is

$$G(i\omega_0) \approx -0.157$$

and the amplitude margin is

$$A_m = \frac{1}{|G(i\omega_0)|} = 6.35.$$

The argument of the system at the frequency ω_c where the magnitude is 1, i.e., where the Nyquist plot intersects the unit circle, is

$$\arg G(i\omega_c) = -116.1^\circ.$$

The phase margin is

$$\varphi_m = 180^\circ + \arg G(i\omega_c) = 63.9^\circ.$$

c. According to the gain and phase margins above, the system has good robustness. However, the Nyquist curve is very close to the critical point, and if the gain and the time-delay were to increase we can easily end up with an unstable closed-loop system.

6 a. The transfer function of the controller is

$$G_r(s) = K(1 + \frac{1}{sT_i}).$$

The transfer function of the closed-loop system is

$$G_{cl} = \frac{G_p G_r}{1 + G_p G_r} = \frac{3K(s + \frac{1}{T_i})}{s^2 + (2 + 3K)s + \frac{3K}{T_i}}$$

The denominator is compared to the desired characteristic polynomial $(s+2)^2 = s^2 + 4s + 4$. This gives $K = \frac{2}{3}$ and $T_i = \frac{1}{2}$.

b. See the course literature.

7 a. In order for the controller to be asymptotically stable $T_f > 0$. The transfer function from the reference to the error is given by

$$G_{er}(s) = \frac{1}{1 + P(s)C(s)} = \frac{\frac{1}{T_f}(s-1)(sT_f + 1)}{s^2 + \frac{1+8K-T_f}{T_f}s + \frac{2K-1}{T_f}}$$

Second order systems are asymptotically stable if and only if all coefficients of the characteristic polynomial are strictly positive i.e.,

$$\frac{2K-1}{T_f} > 0 \quad \Leftrightarrow \quad K > \frac{1}{2}$$
$$\frac{1+8K-T_f}{T_f} > 0 \quad \Leftrightarrow \quad T_f < 8K + 1$$

so engineer C was right.

b. The reference is a unit step i.e., $R(s) = \frac{1}{s}$. To find the stationary error we assume that $K > \frac{1}{2}$ and make use of the final value theorem

$$\lim_{s \to 0} sE(s) = \lim_{s \to 0} sG_{er}(s)R(s) = \lim_{s \to 0} G_{er}(s) = -\frac{1}{2K-1}$$

The magnitude of the error should be less than or equal to 0.05

$$\frac{1}{2K-1} \leq 0.05 \Leftrightarrow K \geq 9.5$$

The smallest K that satisfies the specification is $K = 9.5$.

8 a.

$$G(s) = C(sI - A)^{-1}B = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} s+1 & 1 \\ 1 & s+3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} =$$
$$\frac{2}{s^2 + 4s + 2} \begin{pmatrix} s+3 & -1 \\ -1 & s+1 \end{pmatrix} = \begin{pmatrix} \frac{2(s+3)}{s^2+4s+2} & \frac{-2}{s^2+4s+2} \\ \frac{-2}{s^2+4s+2} & \frac{2(s+1)}{s^2+4s+2} \end{pmatrix}$$

b. Determine the Relative Gain Array (RGA), Λ, of the system.

$$\Lambda = G(0) .* (G(0)^{-1})^T = \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} .* \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix}$$

Elements close to 1 indicate good pairing. In this case, it is best to choose $(1 \leftrightarrow 1)$ och $(2 \leftrightarrow 2)$.

The engineer has made the following mistakes:

1. The transition where the parallel tracks join will never be true. The condition should always be true.
2. The production of fire units is started but never stopped. There are different ways of stopping the production. You could for instance change the line `S ProduceFire = 1;` to `N ProduceFire;`. Another possibility is to add an exit action in the same block that starts the production i.e., add the line `X ProduceFire = 0;`. A third option is to end the production when 50 units are produced by adding `S ProduceFire = 0;` to the step after the transition `FireUnitsDone == 50`.
3. The conditions for when to produce another batch and when to go back to the start should be switched. According to the conditions in the original diagram the system goes back to the initial step if `Batches` is less than ten.
4. The integer `Batches` needs to be reset in the initial step i.e, the line `S Batches = 0;` needs to be added to the initial step.