Systems Engineering/Process control L8

Frequency analysis

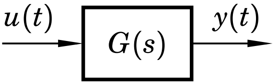
- Frequency response
- Bode- and Nyquist diagram
- Stability and stability margins

Reading: *Systems Engineering and Process Control*: 8.1–8.5

Frequency analysis

- Study how systems react on signals with different frequencies
- Examples:
  - Load disturbances – mostly low frequencies
  - Measurement noise – high frequencies
- If system linear each frequency can be studied separately
  - Sine wave in $\Rightarrow$ sine wave out
  - Can be used to experimentally derive transfer functions

Frequency response



$$u(t) = \sin \omega t$$
$$y(t) = A \sin(\omega t + \varphi)$$

$$A = |G(i\omega)|$$
$$\varphi = \arg G(i\omega)$$

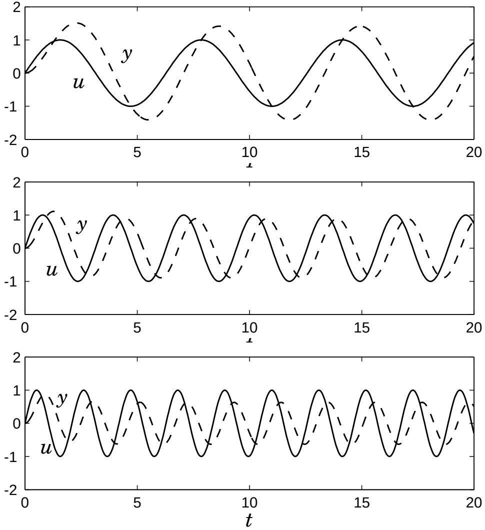
- ω: frequency [rad/s]
- $G(i\omega)$: frequency function
- $|G(i\omega)|$: amplitude (function), amplification, magnitude
- $\arg G(i\omega)$: phase(function), phase shift

Example: $G(s) = \frac{2}{s+1}$

$\omega = 1$ rad/s:

$\omega = 2$ rad/s:

$\omega = 3$ rad/s:



Rules for complex numbers

$$z = x + iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$|z_1 z_2| = |z_1||z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg z = \arctan\frac{y}{x} \quad \text{(if } x > 0\text{)}$$

$$\arg z_1 z_2 = \arg z_1 + \arg z_2, \qquad \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

Example: $G(s) = \frac{2}{s+1}$

$$G(i\omega) = \frac{2}{i\omega + 1}$$

$$|G(i\omega)| = \frac{2}{\sqrt{\omega^2 + 1}}$$

$$\arg G(i\omega) = -\arctan \omega$$

ω	$\lvert G(i\omega)\rvert$	$\arg G(i\omega)$
0	2	0°
1	$\sqrt{2}$	-45°
∞	0	-90°

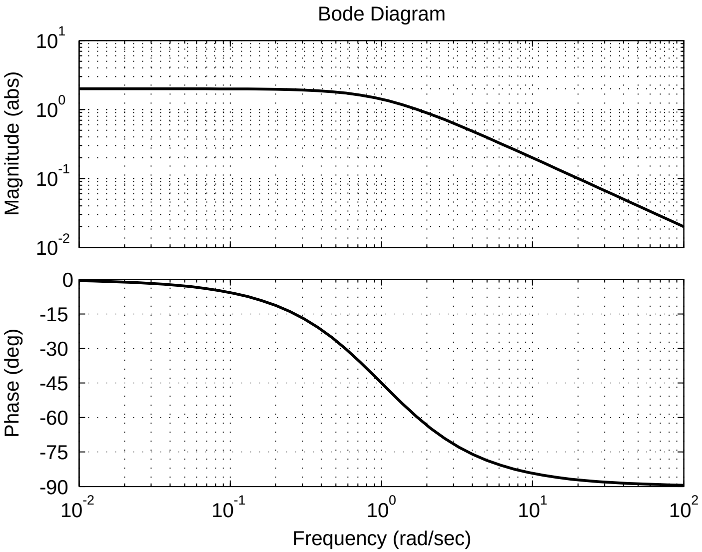
Bode diagrams

Draw $|G(i\omega)|$ and $\arg G(i\omega)$ as functions of ω

- Amplitude curve $|G(i\omega)|$ drawn in log-log-scale
- Phase curve $\arg G(i\omega)$ draw in log-lin-scale

(MATLAB command: `bode`)

Example: $G(s) = \frac{2}{s+1}$

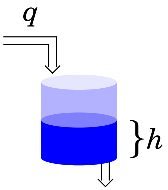


Mini problem

Read from Bode diagram:

- How much are inputs with frequency 0.5 rad/s
  - amplified
  - phase shifted
- How much are inputs with frequency 5 rad/s
  - amplified
  - phase shifted

Example: Level dynamics in tank



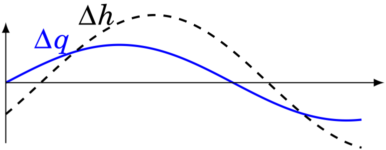
Linearized model:

$$\Delta H(s) = \frac{K_1}{sT_1 + 1}\Delta Q(s)$$

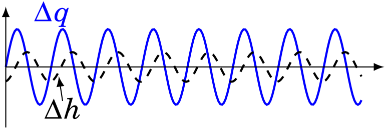
System configured so that $K_1 = 2, T_1 = 1 \Rightarrow$

$$\Delta H(s) = \frac{2}{s + 1}\Delta Q(s)$$

Inflow $\Delta q(t) = \sin 0.5t$:



Outflow $\Delta q(t) = \sin 5t$:



To draw/interpret Bode diagrams

- Suppose $G(s) = G_1(s)G_2(s)G_3(s)\dots$
- Then

$$\begin{aligned} \log|G(i\omega)| &= \log|G_1(i\omega)| + \log|G_2(i\omega)| + \log|G_3(i\omega)| + \dots \\ \arg G(i\omega) &= \arg G_1(i\omega) + \arg G_2(i\omega) + \arg G_3(i\omega) + \dots \end{aligned}$$

- Contribution from G_1, G_2, G_3, … added in both amplitude and phase diagrams

Typical systems

Will show Bode diagrams for the following systems:

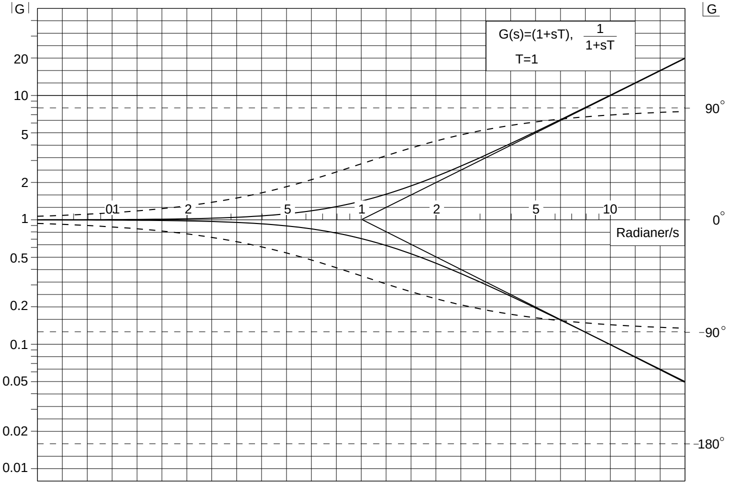
$$\frac{1}{1+sT}, \quad 1+sT$$ real pole, real zero

$$e^{-sL}$$ dead time

$$\frac{\omega_0^2}{s^2+2\zeta\omega_0 s+\omega_0^2}$$ complex poles

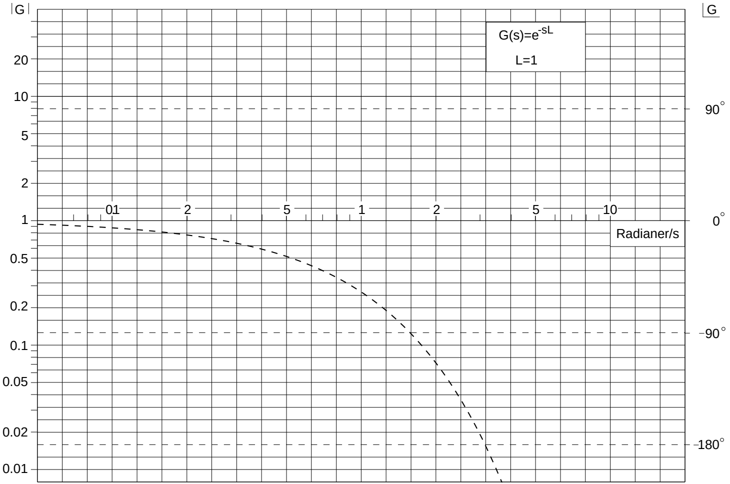
(More examples in book)

Bode diagram for real pole or real zero



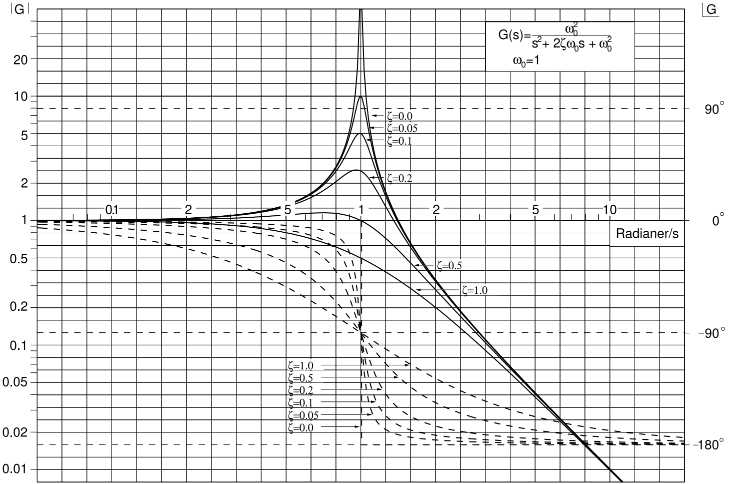
- A pole in $s = -\frac{1}{T}$ bends the amplitude curve down and lowers the phase curve with 90° around $\omega = \frac{1}{T}$; opposite directions for a zero

Bode diagram for dead time



- A dead time lowers phase curve exponentially, does not affect amplitude curve

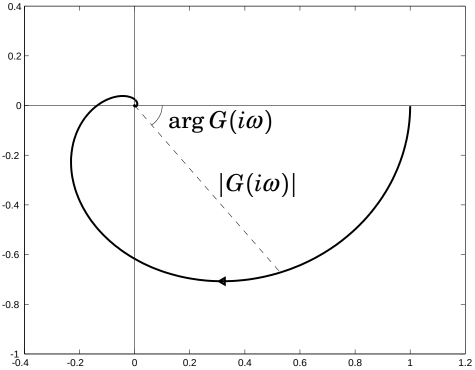
Bode diagram for complex poles



- Complex poles with little damping ζ have big resonance peak at eigen frequency ω_0 in the amplitude curve

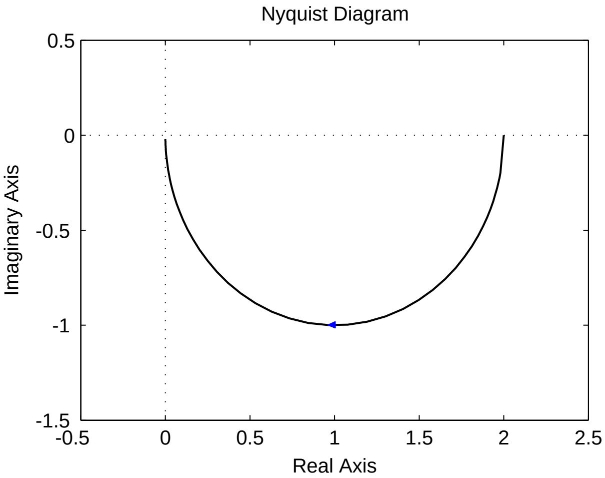
Nyquist diagrams

Draws $G(i\omega)$ as curve in complex plane as ω goes from 0 to ∞

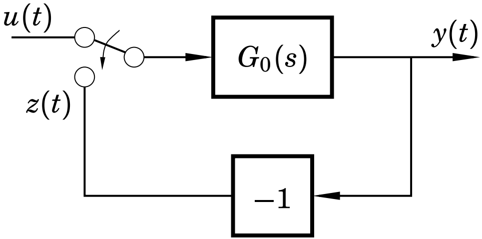


(MATLAB command: `nyquist`)

Example: $G(s) = \frac{2}{s+1}$



Stability for feedback systems



Suppose open-loop system $G_0(s) = G_c(s)G_p(s)$ is stable

$$u(t) = \sin \omega t \quad \Rightarrow \quad \begin{aligned} y(t) &= |G_0(i\omega)| \sin\big(\omega t + \arg G_0(i\omega)\big) \\ z(t) &= -|G_0(i\omega)| \sin\big(\omega t + \arg G_0(i\omega)\big) \\ &= |G_0(i\omega)| \sin\big(\omega t + \arg G_0(i\omega) + 180^\circ\big) \end{aligned}$$

Stability for feedback systems

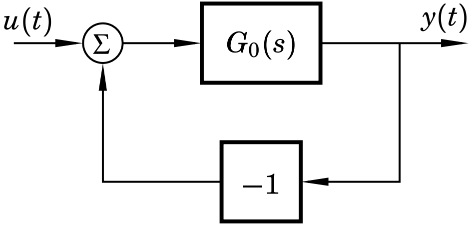
- If $u(t) = z(t)$ a stable self oscillation occurs after switch flip
- This happens if:

$$|G_0(i\omega)| = 1$$
$$\arg G_0(i\omega) = -180^\circ$$

$$\iff$$

Nyquist curve for $G_0(s)$ goes through point -1

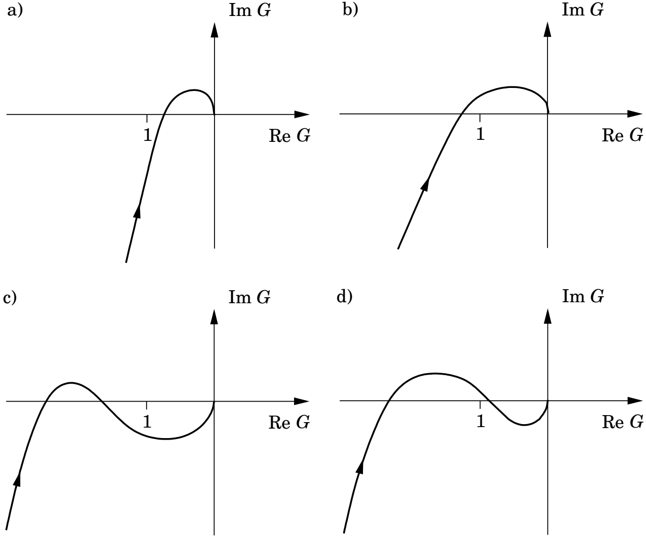
Nyquist's stability theorem



Suppose that the open-loop system $G_0(s)$ has no poles with positive real part. Then the closed-loop system from u to y is asymptotically stable if -1 is to the left of the Nyquist curve of G_0 when going from $\omega = 0$ to $\omega = \infty$.

(Note: Nyquist diagram for G_0 used to infer stability for closed-loop)

Example

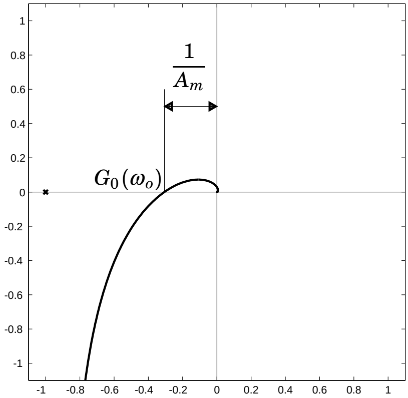


a) as. stable, b) unstable, c) ?, d) ?

Amplitude margin

Amplitude margin shows maximal gain increase before instability:

- Let ω_0 be smallest frequency with $\arg G_0(i\omega_0) = -180°$
- Amplitude margin is given by $A_m = 1/|G_0(i\omega_0)|$

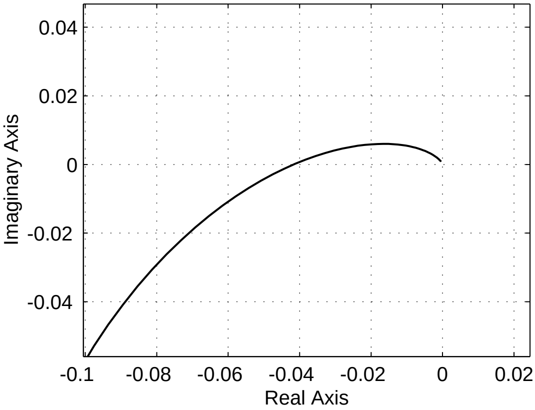


Example (I1 Problem 10)

- Transfer function from input to concentration in compartment 3

$$G_3(s) = \frac{1}{(s+3.732)(s+1)(s+0.2679)}$$

- Nyquist curve:

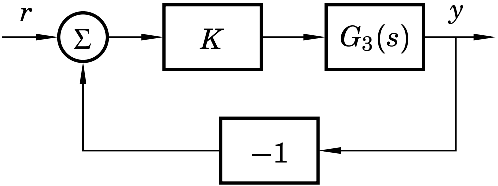


Example (I1 problem 10)

- Read amplitude margin:

$$\frac{1}{A_m} = 0.042 \quad \Leftrightarrow \quad A_m = 24$$

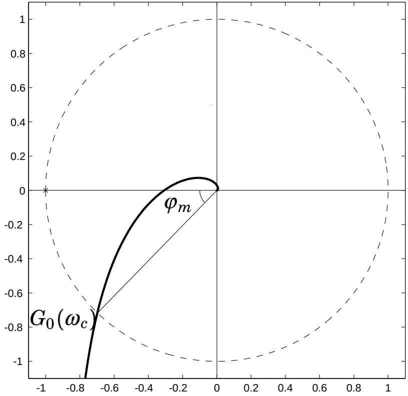
- Interpretation: Maximal gain for P controller is 24 to guarantee a stable closed loop system



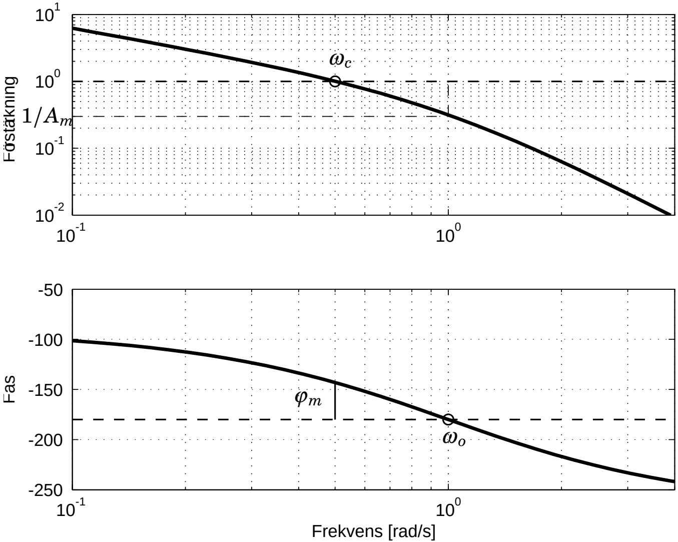
Phase margin

Phase margin shows allowed phase decrease before instability:

- Let ω_c be the smallest frequency with $|G_0(i\omega_c)| = 1$
- Phase margin is given by $\varphi_m = 180° + \arg G_0(i\omega_c)$



Amplitude and phase margin in Bode diagram



Robustness

- To get a robust system we want:
  - $A_m \in [2, 6]$
  - $\varphi_m \in [45°, 60°]$
- The bigger the margins, the less sensitive to model errors