Systems Engineering/Process control L7

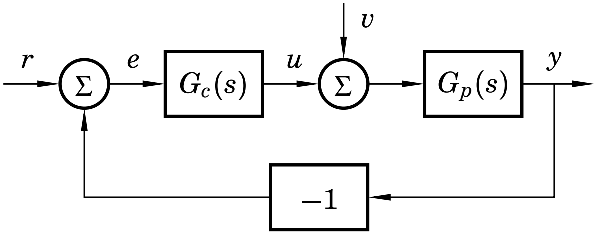
Feedback systems, cont'd

- Analysis of stationary errors
- Feedback linearization
- Sensitivity analysis

Reading: *Systems Engineering and Process Control:* 7.1–7.2

Analysis of stationary errors

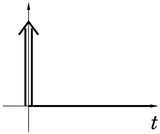
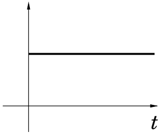
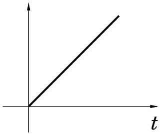
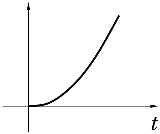
- Standard loop with controller $G_c(s)$, process $G_p(s)$
- Suppose closed-loop system is stable



What is stationary error $e(\infty)$ for given

- reference r (servo problem)?
- load disturbances v (control problem)?

Signal models

		Load disturbance	Reference
Impulse:		$V(s) = 1$	—
Step:		$V(s) = \frac{a}{s}$	$R(s) = \frac{a}{s}$
Ramp:		$V(s) = \frac{b}{s^2}$	$R(s) = \frac{b}{s^2}$
Parabola:		$V(s) = \frac{c}{s^3}$	$R(s) = \frac{c}{s^3}$

Analysis of stationary errors

Control error is given by

$$E(s) = \frac{1}{1 + G_p(s)G_c(s)}R(s) - \frac{G_p(s)}{1 + G_p(s)G_c(s)}V(s)$$

The stationary error can be computed using end-value theorem:

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Stationary error – Servo problem

- Let $V(s) = 0$ and suppose

$$G_p(s)G_c(s) = \frac{KQ(s)}{s^n P(s)}, \quad Q(0) = P(0) = 1$$

($n =$ is total number of integrators in controller and process)

- Then:

$$E(s) = \frac{s^n P(s)}{s^n P(s) + KQ(s)} R(s)$$

Stationary error – Servo problem

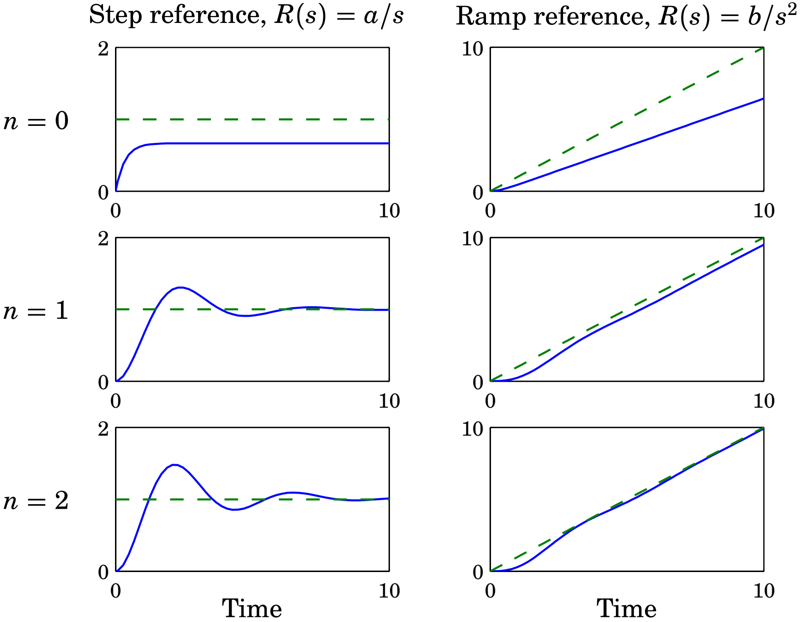
- Stationary error with step reference, $R(s) = a/s$:

$$e(\infty) = \lim_{s\to 0} \frac{s^n P(s) a}{s^n P(s) + KQ(s)} = \begin{cases} a/(1+K) & n = 0 \\ 0 & n \geq 1 \end{cases}$$

- Stationary error with ramp reference, $R(s) = b/s^2$:

$$e(\infty) = \lim_{s\to 0} \frac{s^n P(s)}{s^n P(s) + KQ(s)} \cdot \frac{b}{s} = \begin{cases} \infty & n = 0 \\ b/K & n = 1 \\ 0 & n \geq 2 \end{cases}$$

Stationary error – Servo problem



Stationary error – Control problem

Let $R(s) = 0$ and suppose $K = K_1 K_2$, $Q = Q_1 Q_2$, $P = P_1 P_2$ so that

$$G_c(s) = \frac{K_1 Q_1(s)}{s^m P_1(s)}, \quad Q_1(0) = P_1(0) = 1$$

$$G_p(s) = \frac{K_2 Q_2(s)}{s^{n-m} P_2(s)}, \quad Q_2(0) = P_2(0) = 1$$

- $m =$ number of integrators in controller
- $n =$ total number of integrators in controller and process

Then:

$$E(s) = -\frac{s^m K_2 P_1(s) Q_2(s)}{s^n P(s) + K Q(s)} V(s)$$

Stationary error – Control problem

- Stationary error with impulse disturbance, $V(s) = 1$:

$$e(\infty) = 0$$

- Stationary error with step disturbance, $V(s) = a/s$:

$$e(\infty) = \lim_{s\to 0} -\frac{s^m K_2 P_1(s) Q_2(s) a}{s^n P(s) + K Q(s)} = \begin{cases} -aK_2/(1+K) & m = 0,\, n = 0 \\ -a/K_1 & m = 0,\, n \geq 1 \\ 0 & m \geq 1 \end{cases}$$

- Stationary error with ramp disturbance, $V(s) = b/s^2$:

$$e(\infty) = \lim_{s\to 0} -\frac{s^m K_2 P_1(s) Q_2(s)}{s^n P(s) + K Q(s)} \cdot \frac{b}{s} = \begin{cases} -\infty & m = 0 \\ -b/K_1 & m = 1 \\ 0 & m \geq 2 \end{cases}$$

Stationary error – Conclusions

Servo problem: Follow ...

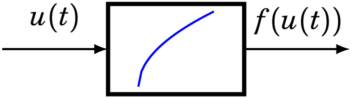
- step reference requires 1 integrator in controller/process
- ramp reference requires 2 integrators in controller/process
- parabola reference requires 3 integrators in controller/process
- ...

Control problem: Eliminate ...

- impulse disturbance requires as. stable closed-loop system
- step disturbance requires 1 integrator in controller
- ramp disturbance requires 2 integrators in controller
- ...

Feedback linearization

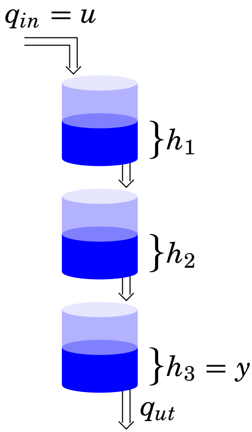
Sensor (valves, pumps,...) nonlinearities complicate control and analysis



Two methods to linearize static nonlinearity $f(u)$:

- Pre-multiply with inverse nonlinearity
- Use (inner) feedback (often P control)

Example: Control of triple tank



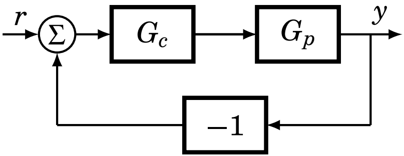
Transfer function from u to y:

$$G_p(s) = \frac{2}{(s+1)^3}$$

PI controller:

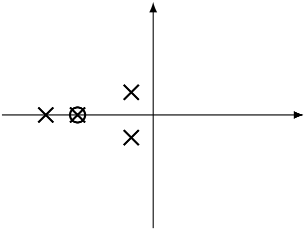
$$G_c(s) = \frac{s+1}{8s}$$

Block diagram:

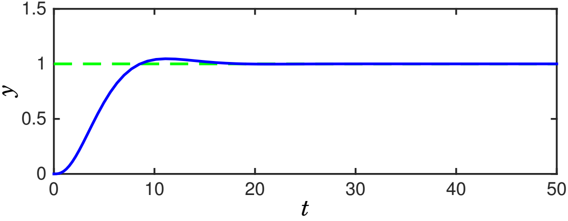


Example: Control of triple tank

- Singularity diagram for closed-loop system:

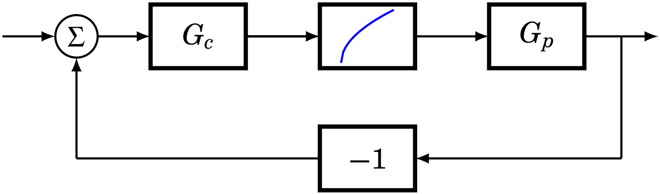


- Step response for feedback system:



Example: Control of triple tank

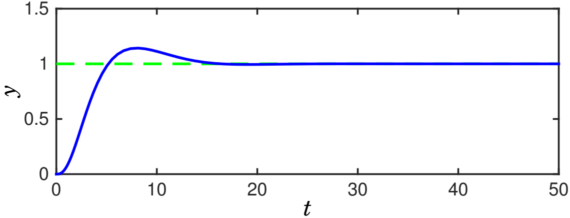
- Suppose inflow through fast opening valve
- Valve nonlinear characteristics: $f(u) = \sqrt{u},\ 0 \leq u \leq 1$:



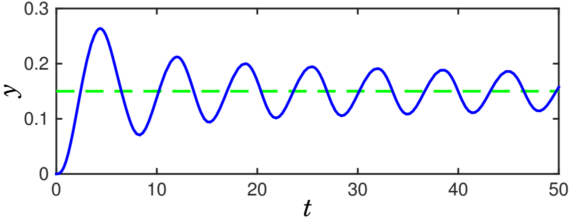
- How is step response affected by nonlinearity?

Example: Control of triple tank

- Step response for closed-loop system, $r(t) = 1$:



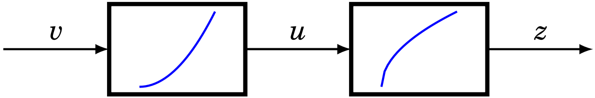
- Step response for closed-loop system, $r(t) = 0.15$:



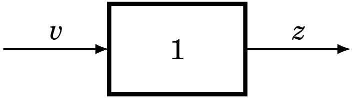
- Nonlinearity gives different step responses for different step sizes!

Linearization using inverse nonlinearity

- Pre-multiply signal to valve with $g(v) = v^2$:



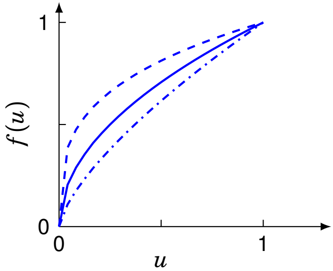
- Then $z = f(u) = f(g(v)) = f(v^2) = \sqrt{v^2} = v$

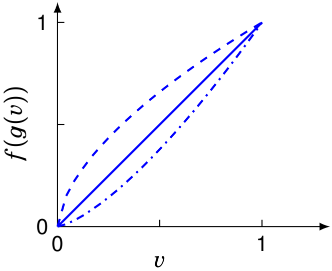


Sensitivity to parameter variations

- If valve true characteristic is (left figure)

$$z = f(u) = u^{\alpha}, \quad \alpha = 0.3,\, 0.5,\, 0.7$$



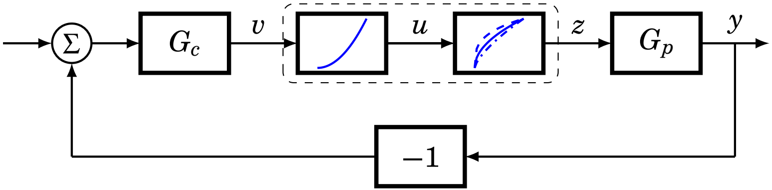


- the following compensation with $g(v) = v^2$ is achieved (right figure):

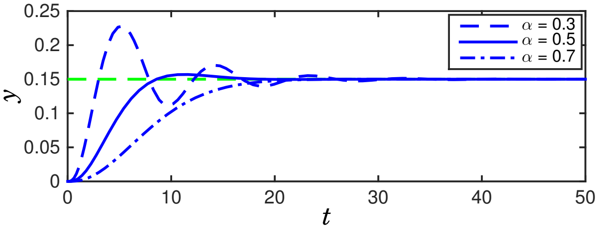
$$\begin{aligned} \alpha = 0.3: &\quad f(g(v)) = f(v^2) = v^{0.6} \\ \alpha = 0.5: &\quad f(g(v)) = f(v^2) = v \\ \alpha = 0.7: &\quad f(g(v)) = f(v^2) = v^{1.4} \end{aligned}$$

Static compensation of nonlinearity

- Static compensation $g(v) = v^2$

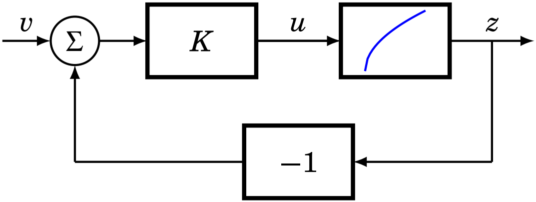


- Step response for closed-loop system, $r(t) = 0.15$:



Feedback linearization of static nonlinearity

- Measure flow and introduce P control around valve:



P controller: $u = K(v - z) \quad \Rightarrow$

Nonlinearity v to z:

$$z = f(u) = u^{0.5} = \sqrt{K(v - z)}$$

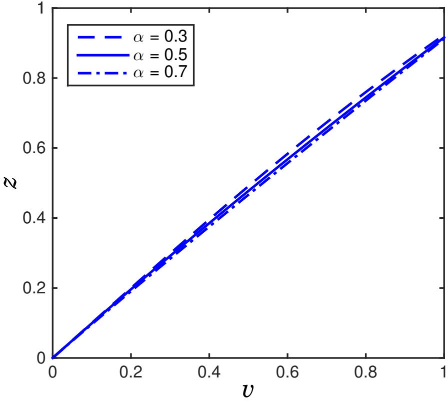
$$z^2 = K(v - z)$$

$$v = z + \frac{z^2}{K}$$

- If K big we get $z \approx v$

Feedback linearization of static nonlinearity

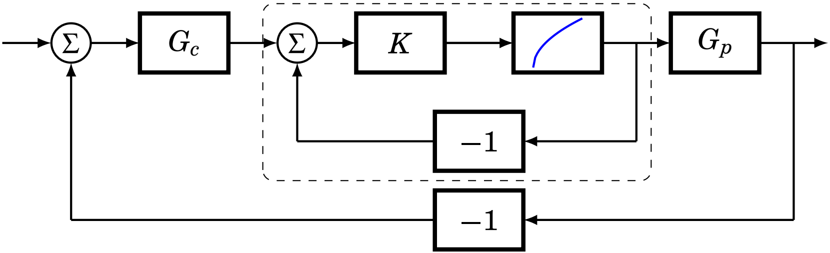
- Relation between v and z for $K = 10$ and different $f(u) = u^{\alpha}$:



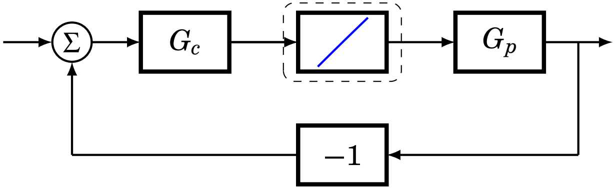
- Close to linear and insensitive to parameter variations!
- Many (static) nonlinearities can be linearized the same way

Triple tank: Compensation of nonlinearity

- Compensate nonlinearity with inner feedback loop:

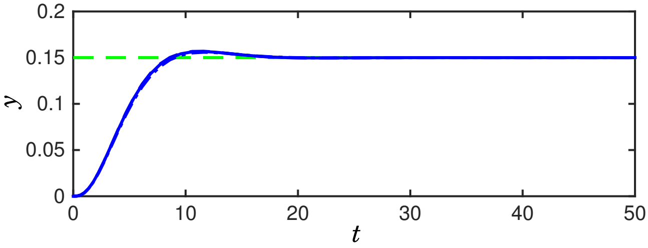


- This gives:



Step response for compensated system

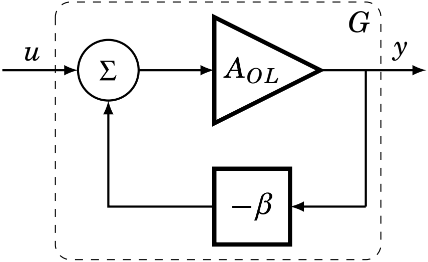
- Step response for closed-loop system
  - reference $r(t) = 0.15$
  - true nonlinearity $f(u) = u^{\alpha}$ with $\alpha = 0.3, 0.5, 0.7$:



- Almost completely insensitive to parameter variations

Example: Feedback amplifier

- Long distance phone calls: Many amplifiers needed
- Historic amplifiers A_{OL} distorted sound (nonlinear amplification)
- Feedback amplifier invented by H. Black 1927



- Gain from u to y: $G = \dfrac{A_{OL}}{1 + \beta A_{OL}}$
- If $\beta A_{OL} \gg 1 \Rightarrow G \approx 1/\beta$
  (small β gives big amplification, requires A_{OL} big)

Example: Feedback amplifier

- Feedback can eliminate frequency variations in amplification

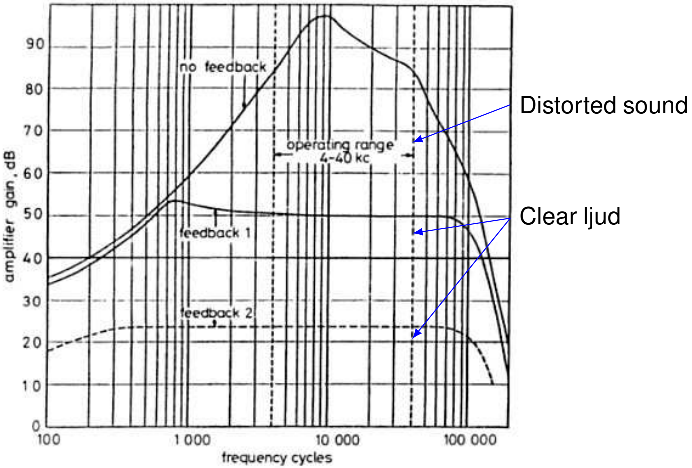


Figure 3.3 *Gain frequency characteristics with and without feedback*

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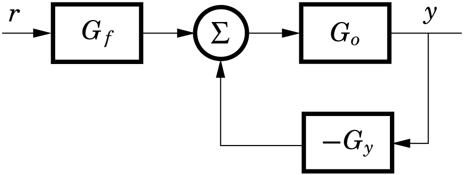
Sensitivity analysis



- G_o = open-loop system (often controller and process)
- G_f = feedforward
- G_y = feedback (often -1)

Transfer function of closed-loop system from r to y:

$$G_c(s) = \frac{G_f(s)G_o(s)}{1 + G_o(s)G_y(s)}$$

How is G_c affected by variations in components G_f, G_o, G_y?

Sensitivity analysis

- Define the **relative sensitivity** of a transfer function G w.r.t. component H as

$$S_H = \frac{dG}{dH} \cdot \frac{H}{G} = \frac{dG}{G} \bigg/ \frac{dH}{H}$$

- For G_c we have:

$$S_{G_f} = 1$$

$$\boxed{S_{G_o} = \frac{1}{1 + G_o G_y}}$$

$$S_{G_y} = -\frac{G_o G_y}{1 + G_o G_y}$$

Sensitivity analysis

- Relative sensitivity S_{G_o} small if gain $G_o G_y$ big
- Too high feedback gain may cause:
  - much measurement noise to be fed back to system
  - instability
- Typical design compromise: Want G_0 big at low frequencies (integral action), small at higher frequencies

Feedback system – Summary

Pros:

- Changed dynamics
  - Faster, more well damped, etc
  - Closed-loop poles decided by controller parameters – pole placement
- Elimination of disturbances
  - Elimination of stationary error requires integral action in controller
- Reduced sensitivity to process variation and nonlinearities

Cons:

- Measurement noise is fed back to process
- Can lead to instability