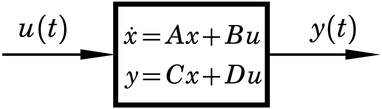
Systems Engineering/Process control L5

- Impulse and step response
- Connection between transfer function and step response
- Nonlinear systems

Reading: *Systems Engineering and Process Control*: 5.1–5.3

LTI systems – repetition (L3–L4)

State-space model

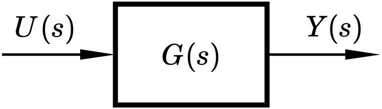


System response:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Stability:
Decided by eigenvalues of A

Input-output model



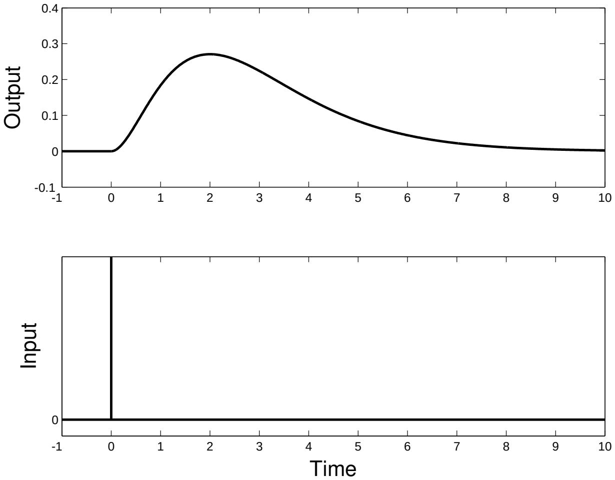
System response:

$$Y(s) = G(s)U(s)$$

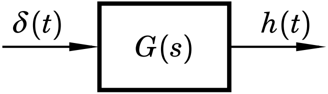
Stability:
Decided by poles to $G(s)$

Impulse response

- Suppose that the system is in equilibrium
- How does output react to input impulse (Dirac function)?



Impulse response for linear systems



1. Laplace transform input: $U(s) = 1$
2. Output becomes:

$$H(s) = G(s)U(s) = G(s)$$

3. Inverse transform gives impulse response:

$$h(t) = \mathcal{L}^{-1}\{G(s)\}$$

$h(t)$ also called weighting function

Impulse response for linear systems

For a system on state-space for the impulse response becomes:

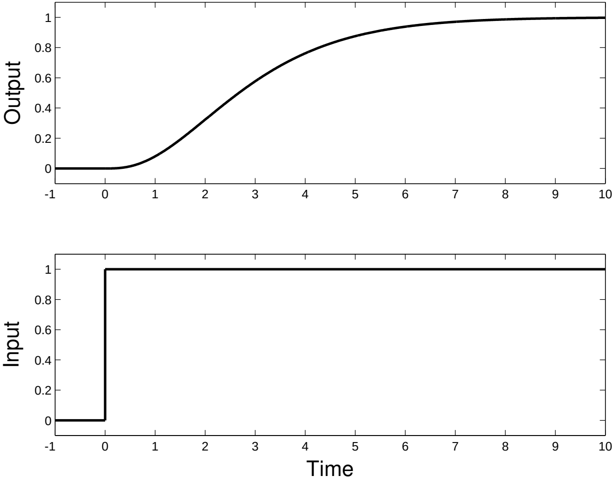
$$h(t) = Ce^{At}B + D\delta(t)$$

Stability notions (again):

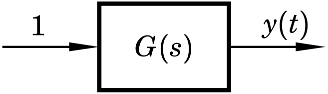
- $h(t)$ limited (except maybe at $t = 0$) $\Longleftrightarrow$ stable system
- $h(t) \to 0 \Longleftrightarrow$ asymptotically stable system
- $h(t)$ unlimited $\Longleftrightarrow$ unstable system

Step response

- Suppose system in equilibrium
- How does the output change after step in input?



Step response for linear systems



1. Laplace transform input: $U(s) = \frac{1}{s}$
2. Output becomes:

$$Y(s) = G(s)U(s) = G(s)\frac{1}{s}$$

3. Inverse transformation gives step response:

$$y(t) = \mathcal{L}^{-1}\left\{G(s)\frac{1}{s}\right\} = \int_0^t h(\tau)d\tau$$

(The step response is the integral of the impulse response)

Static gain

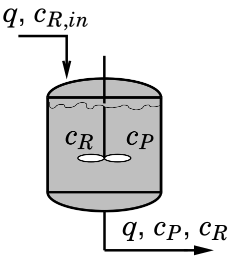
- Step response end value is called **static gain** of system
- Can be computed using the end value theorem:

$$Y(s) = G(s)\frac{1}{s}$$

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sG(s)\frac{1}{s} = G(0)$$

- Note: Step response end value exists only for asymptotically stable systems!

Example: CSTR



- Volume V, flow q
- Reaction $R \to P$
- Reaction rate k

Transfer function from $c_{R,in}$ to c_P:

$$G(s) = \frac{\frac{q}{V}k}{(s + \frac{q}{V} + k)(s + \frac{q}{V})}$$

Static gain:

$$G(0) = \frac{\frac{q}{V}k}{(\frac{q}{V} + k)\frac{q}{V}} = \frac{1}{\frac{q}{kV} + 1}$$

Interpretation: If $c_{R,in}$ increases with 1, c_P increases with $\frac{1}{\frac{q}{kV}+1}$ at equilibrium

Connection between transfer fcn and step response

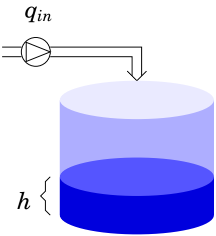
System type:

- Integrator
- First order system
- Second order systems with real poles
- Second order systems with complex poles
- Systems with one zero
- Systems with time delays

Integrating systems

$$G(s) = \frac{K}{s}$$

Example: Tank without free outflow:



- Cross-sectional area: A

Transfer function from q_{in} to h:

$$G(s) = \frac{1/A}{s}$$

Integrating systems

- Pole:

$$s = 0$$

- Step response:

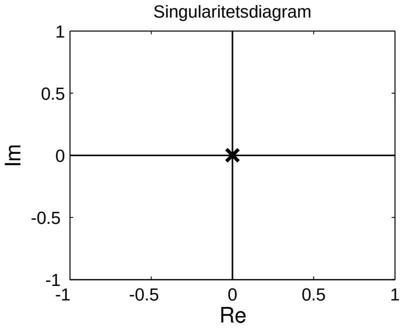
$$Y(s) = G(s)\frac{1}{s} = \frac{K}{s^2}$$

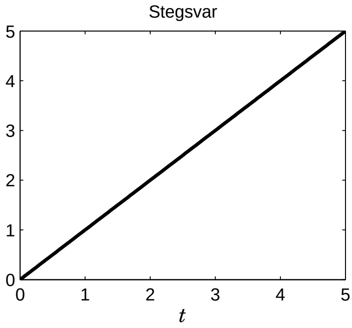
$$y(t) = Kt$$

- No end value, since system is not asymptotically stable

Integrating systems

$K = 1$:

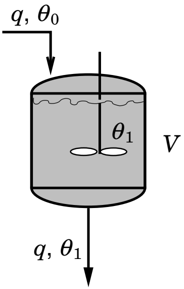




1st order systems

$$G(s) = \frac{K}{1 + sT}, \quad T > 0$$

Example: Temperature dynamics in a tank:



Transfer function from θ_0 to θ_1:

$$G(s) = \frac{1}{1 + s\frac{V}{q}}$$

1:a order systems

- Pole:

$$s = -1/T$$

- Step response:

$$Y(s) = G(s)\frac{1}{s} = \frac{K}{s(1+sT)}$$

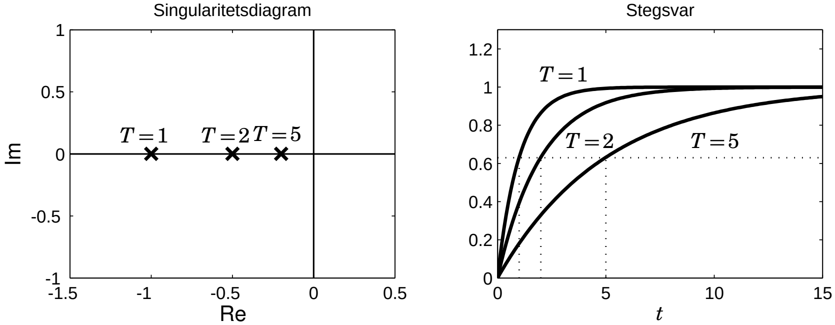
$$y(t) = K\left(1 - e^{-t/T}\right)$$

- T is called **time constant** of the system

$$y(T) = (1 - e^{-1})K \approx 0.63K$$

1:a order systems

$K = 1$:

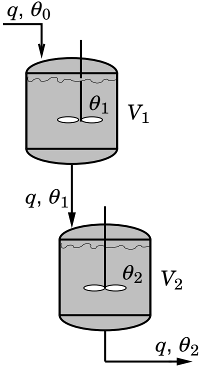


- Step response speed decided by distance from pole to origin

2nd order systems with real poles

$$G(s) = \frac{K}{(1+sT_1)(1+sT_2)}, \qquad T_1, T_2 > 0$$

Example: Temperature dynamics in coupled tanks:



Transfer function from θ_0 to θ_2:

$$G(s) = \frac{1}{\left(1+s\frac{V_1}{q}\right)\left(1+s\frac{V_2}{q}\right)}$$

2:a order systems with real poles

- Poles:

$$s = -1/T_1, \quad s = -1/T_2$$

- Step response:

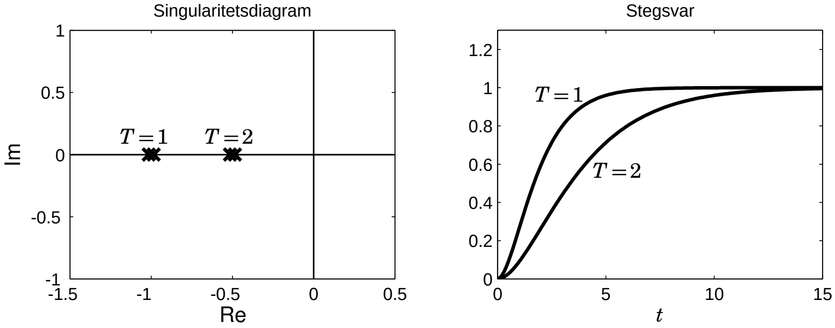
$$Y(s) = G(s)\frac{1}{s} = \frac{K}{s(1+sT_1)(1+sT_2)}$$

$$y(t) = \begin{cases} K\left(1 - \frac{T_1 e^{-t/T_1} - T_2 e^{-t/T_2}}{T_1 - T_2}\right), & T_1 \neq T_2 \\ K\left(1 - e^{-t/T} - \frac{t}{T}e^{-t/T}\right), & T_1 = T_2 = T \end{cases}$$

- Two time constants: T_1, T_2

2nd order systems with real poles

$K = 1$:



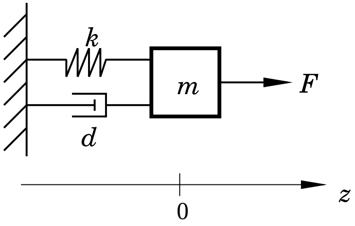
- Two poles gives softer and slower response than single pole
  - equivalent time constant: $T_{eq} = T_1 + T_2$
- If $T_1 \gg T_2$ system behaves essentially as 1st order system with time constant T_1

2nd order systems with complex poles

$$G(s) = \frac{K\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}, \qquad \omega_0 > 0,\ 0 < \zeta < 1$$

- ω_0 = undamped frequency
- ζ = relative damping

Example: Position dynamics for mechanical system



Transfer function from F to z:

$$G(s) = \frac{\frac{1}{m}}{s^2 + \frac{d}{m}s + \frac{k}{m}}$$

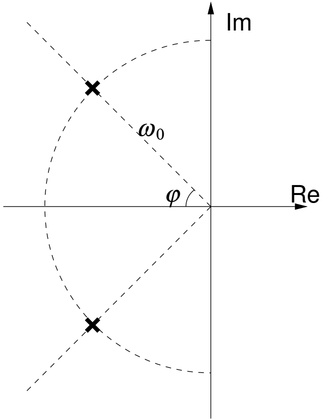
Complex poles if $d < 2\sqrt{km}$

2nd order systems with complex poles

Poles:

$$s = -\zeta\omega_0 \pm i\sqrt{1-\zeta^2}\,\omega_0$$

$$\zeta = \cos\varphi$$

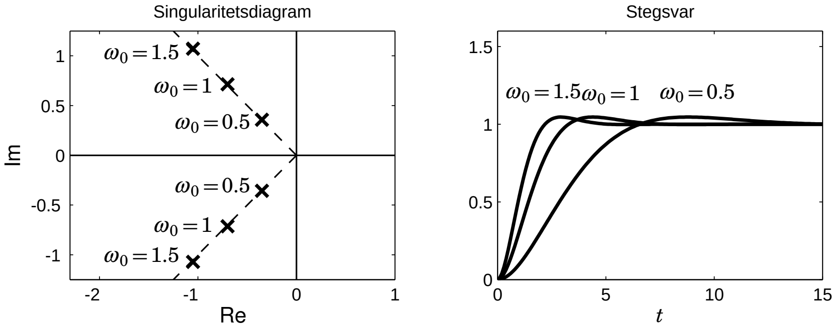


Step response:

$$y(t) = K\left(1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_0 t}\sin\left(\omega_0\sqrt{1-\zeta^2}\,t + \arccos\zeta\right)\right)$$

2nd order systems with complex poles

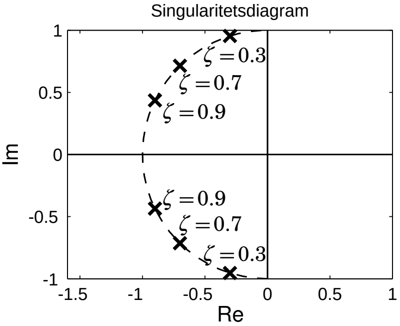
$K = 1$:

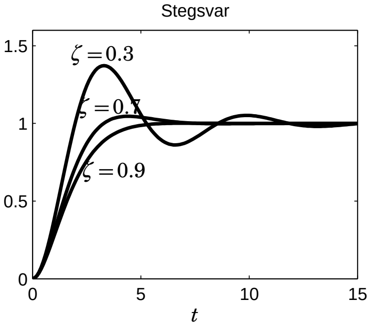


- System speed decided by distance from poles to the origin

2nd order systems with complex poles

$K = 1$:





- System damping decided by angle of the poles

Systems with zeros

- Suppose the system is given by

$$(1 + T_z s)G_0(s)$$

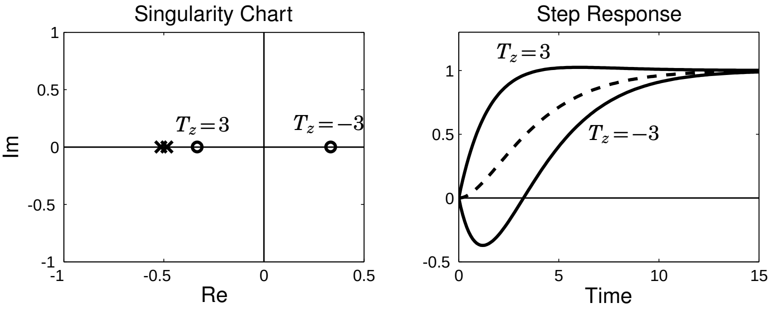
- Zero in $s = -\frac{1}{T_z}$
- Step response:

$$y(t) = \mathcal{L}^{-1}\left\{G_0(s)\frac{1}{s}\right\} + T_z\mathcal{L}^{-1}\{G_0(s)\}$$

  - Weighted sum of impulse response and step response for $G_0(s)$
  - Big impact if zero close to the origin (T_z large)

2nd order systems with zeros

Example: $G(s) = \dfrac{1 + sT_z}{(1 + 2s)^2}$



Dashed step response for $G_0(s) = \dfrac{1}{(1 + 2s)^2}$

- Zeros affect initial response
- R.h.p. zeros gives inverse response behavior initially

Systems with time delay

- Suppose the system is given by:

$$G(s) = G_0(s)e^{-sL}, \quad L > 0$$

- Step response for part without delay $G_0(s)$:

$$y_0(t) = \mathcal{L}^{-1}\left\{G_0(s)\frac{1}{s}\right\}$$

- Step response with time delay:

$$y(t) = y_0(t - L)$$

(e^{-sL} cannot be interpreted with (finitely many) poles and zeros)

Interpretation of poles and zeros

Poles

- Depends only on A-matrix, e.g., on system inner dynamics
- Decides system:
  - stability
  - speed
  - damping

Zeros

- Harder to interpret
- Depends on how inputs and outputs are connected to system (i.e., depends on B, C, and D matrices)
- A zero in $s = a$ cancels the signal e^{at}
- Influences mostly the initial step response behavior

Processes that are difficult to control

Processes with:

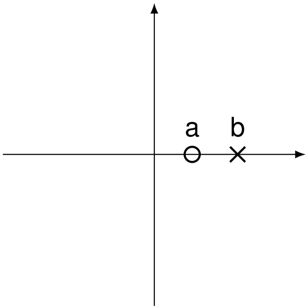
- Poles in right half-plane (unstable)
  - The bigger the real part (> 0) the harder to control
- Zeros in right half-plane (reversed response initially)
  - The smaller real part (> 0) the harder to control
- Time delays
  - The longer time delay, the harder to control

Processes that are impossible to control

- Systems with poles and zeros in right half-plane $(a, b > 0)$:

$$G(s) = \frac{Q(s)(s-a)}{P(s)(s-b)}$$

- If $a = b$: impossible to control
- If $a \leq 3b$: impossible to control in practice



Examples

Bicycle with back wheel steering
$a/b \approx 0.7$ at 1 m/s

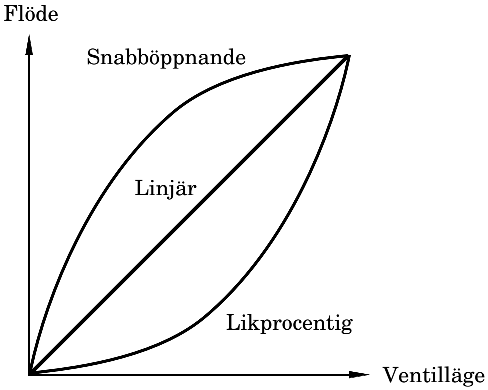
X29, $a/b \approx 4.33$

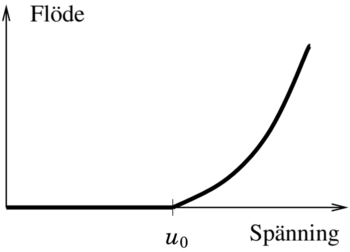
Nonlinear systems

Different kinds of nonlinearities:

- Nonlinearities in actuators and sensors, e.g.,:
  - upper and lower limits on actuators and sensors
  - pumps and valves with nonlinear characteristics
  - friction and dead zones
  - nonlinear sensors for temperature, flow, concentration
- Nonlinear dynamics in the process, e.g.,:
  - level dependent outflow speed in a tank
  - temperature dependent reaction speed in reactor
  - population dependent rate of growth
- Nonlinearities in the controller, e.g.,:
  - on/off control

Example: Valve and pump characteristics



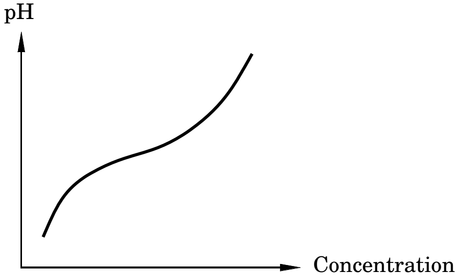


Methods to compensate for nonlinearity:

- Compensate with table/mathematical function
- Feedback around static nonlinearity (better and more robust)

Example: pH control

Want to control pH but measures concentration:

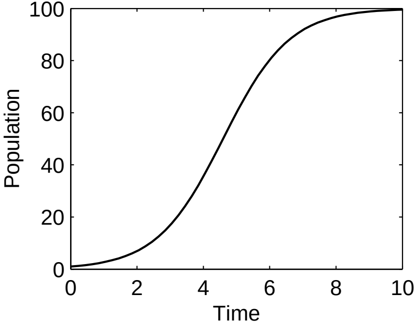


- Can be compensated for with table/mathematical function

Example: Logistic growth model

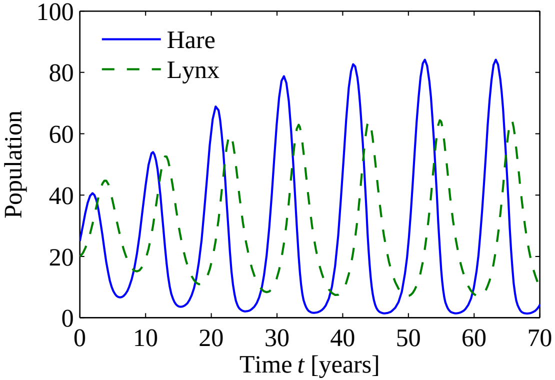
$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right)$$

$x =$ population, $r =$ net growth rate, $k =$ carrying capacity



Example: Hares and Lynxes

$$\frac{dH}{dt} = rH\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H}, \qquad H \geq 0,$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0$$



Linear vs nonlinear systems

Linear systems

- can equivalently be described with linear differential equation, state-space model, transfer function, impulse response, or step response
- are always in equilibrium at $(x, u) = 0$
- global analysis – poles/zeros decide stability globally

Nonlinear systems

- described by nonlinear differential equation/state-space model
- can have many equilibria (stable/unstable) and limit cycles
- simulation, local analysis using linearization