Systems Engineering/Process Control L4

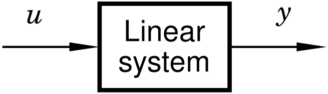
Input-output models

- Laplace transform
- Transfer functions
- Block diagram algebra

Reading: *Systems Engineering and Process Control*: 4.1–4.4

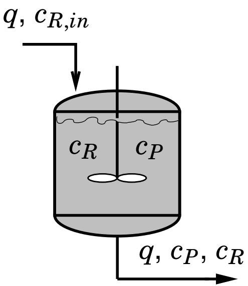
Laplace transform

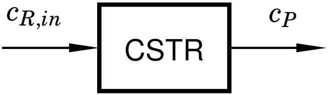
- Powerful mathematical tool to study and solve *linear differential equations*
- Example:



What is output $y(t)$ given a specific input $u(t)$?

Example: CSTR





- Constant volume V and flow q
- First order reaction $R \rightarrow P$
- Reaction rate $r_R = -r_P = -kc_R$

Assume system is in equilibrium.
How does c_P respond to unit step changes in $c_{R,in}$?

Example: CSTR

Mass balance:

$$\text{In} + \text{Prod} = \text{Out} + \text{Acc}$$

$$qc_{R,in} + Vr_R = qc_R + V\frac{dc_R}{dt}$$

$$Vr_P = qc_P + V\frac{dc_P}{dt}$$

Second order state-space model:

$$\frac{dc_R}{dt} = -\left(\frac{q}{V} + k_1\right) c_R + \frac{q}{V} c_{R,in}$$

$$\frac{dc_P}{dt} = k_1 c_R - \frac{q}{V} c_P$$

How to solve equations with Laplace transform?

Laplace transform

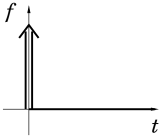
Transforms a function $f(t)$ to another function $F(s)$.

- $f(t)$ is a function of time $t \geq 0$
- $F(s)$ is a function of the "complex frequency" s

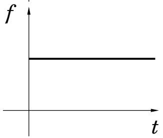
Definition:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

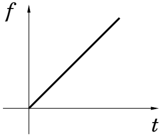
Some common functions (signals)



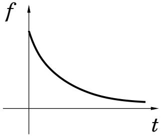
$f(t) = \delta(t)$ Impulse



$f(t) = 1$ Step (constant signal)



$f(t) = t$ Ramp



$f(t) = e^{-at}$ Exponential function

The impulse function

- $\delta(t)$ = impulse at time 0
- Also called Dirac function
- Infinitely high and infinitely thin, but with area 1

Example:

$$\frac{dV}{dt} = \delta(t)$$

Interpretation: “Injection of a unit volume at time 0”

Laplace transform of some common functions

Impulse:

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t) e^{-st} dt = 1$$

Step:

$$\mathcal{L}\{1\} = \int_0^\infty 1 \cdot e^{-st} dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

Exponential function:

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-at} \cdot e^{-st} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{1}{s+a}$$

Excerpt from *collection of formulae* p. 6:

	Laplace transform $F(s)$	Time function $f(t)$	
1	1	$\delta(t)$	Dirac function
2	$\frac{1}{s}$	1	Step function
3	$\frac{1}{s^2}$	t	Ramp function
⋮			
6	$\frac{1}{s+a}$	e^{-at}	

Some properties of the Laplace transform

Excerpt from *collection of formulae* p. 5:

	Laplace transform $F(s)$	Time function $f(t)$	
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Linearity
8	$sF(s) - f(0)$	$f'(t)$	Derivation i t-planet
12	$\frac{1}{s} F(s)$	$\int_0^t f(\tau)\, d\tau$	Integration i t-planet

More useful properties

Excerpt from *collection of formulae* p. 5:

	Laplace transform $F(s)$	Time function $f(t)$	
3	$e^{-as}F(s)$	$\begin{cases} f(t-a) & t-a>0 \\ 0 & t-a<0 \end{cases}$	Time delay
14	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	End point theorem

Compute system response using Laplace transform

1. Laplace transform all terms in the differential equation
   - Use collection of formulae
2. Solve for signal $Y(s)$
3. Use inverse Laplace transform to find $y(t)$
   - Divide into partial fractions first, if needed
   - Use collection of formulae

Example 1

Solve

$$\dot{y} = -3y$$

with initial state $y(0) = 5$.

1. Laplace transform:

$$sY(s) - 5 = -3Y(s)$$

2. Solve for $Y(s)$:

$$(s+3)Y(s) = 5$$

$$Y(s) = \frac{5}{s+3}$$

3. Inverse Laplace (transform nbr. 6):

$$y(t) = 5e^{-3t}$$

Example 2: CSTR

Assume $q = V = k = 1$, $c_R(0) = c_P(0) = 0$, $c_{R,in} = 1$ (step fcn):

$$\dot{c}_R = -2c_R + c_{R,in}$$
$$\dot{c}_P = c_R - c_P$$

1. Laplace transform:

$$sC_R(s) = -2C_R(s) + C_{R,in}(s)$$
$$sC_P(s) = C_R(s) - C_P(s)$$

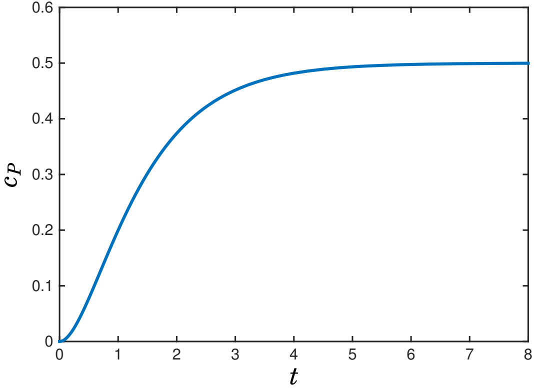
2. Solve for $C_P(s)$:

$$C_R(s) = \frac{1}{(s+2)}C_{R,in}(s)$$
$$C_P(s) = \frac{1}{s+1}C_R(s) = \frac{1}{(s+1)(s+2)}C_{R,in}(s) = \frac{1}{(s+1)(s+2)s}$$

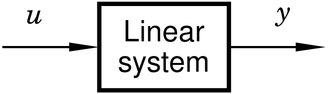
Example 2: CSTR

3. Inverse Laplace (transform nbr. 24):

$$c_P(t) = \frac{1}{2}\left(1 + e^{-2t} - 2e^{-t}\right)$$



Transfer function



- Assume that all initial states are zero
- After Laplace transform, input-output relation can be written:

$$Y(s) = G(s)U(s)$$

$G(s)$ is called the system **transfer function**

Poles and zeros

Often transfer function is be written as:

$$G(s) = \frac{Q(s)}{P(s)}, \quad \deg Q \leq \deg P$$

where $Q(s)$ and $P(s)$ are polynomials

Zeros: roots to $Q(s) = 0$

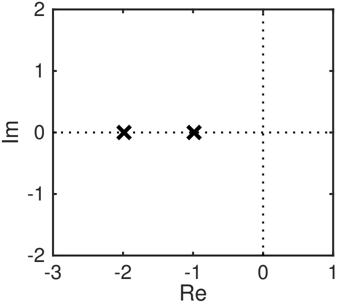
Poles: roots to $P(s) = 0$ (characteristic equation)

Can be drawn in singularity diagram/pole-zero-diagram

- Poles: x
- Zeros: o

Example: CSTR

- *Input-output model*: $C_P(s) = \frac{1}{(s+1)(s+2)} C_{R,in}(s)$
- *Transfer function*: $G(s) = \dfrac{1}{(s+1)(s+2)}$
- *Zeros*: $1 = 0$ has no solutions
- *Poles*: $(s+1)(s+2) = 0$ has solutions $s_1 = -1$, $s_2 = -2$
- *Singularity diagram*:



Connection state-space form–transfer function

- Linear time invariant system on state-space form

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu \\ y &= Cx + Du\end{aligned}$$

- Assume all initial states are zero: $x(0) = 0$
- Laplace transform:

$$\begin{aligned}sX(s) &= AX(s) + BU(s) \\ Y(s) &= CX(s) + DU(s)\end{aligned}$$

Connection state-space form–transfer function

- Solve for $X(s)$:

$$\begin{aligned}(sI - A)X(s) &= BU(s) \\ X(s) &= (sI - A)^{-1}BU(s)\end{aligned}$$

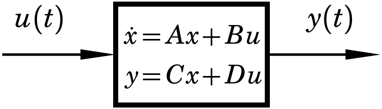
- Insert into equation for $Y(s)$:

$$\begin{aligned}Y(s) &= C(sI - A)^{-1}BU(s) + DU(s) \\ &= \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{G(s)} U(s)\end{aligned}$$

  - Denominator to $G(s)$ given by $\det(sI - A)$
  - Poles to $G(s) \iff$ eigenvalues of A

Comparison

State-space model



System response:

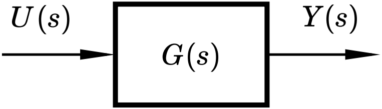
$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$y(t) = Cx(t) + Du(t)$$

Stability:

Decided by eigenvalues to A.

Input-output-model



System response:

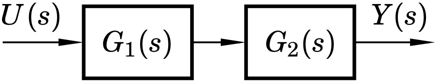
$$Y(s) = G(s)U(s)$$

Stability:

Decided by poles to $G(s)$.

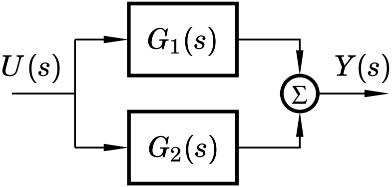
Block diagram computations with transfer functions

Serial connection:



$$Y(s) = G_2(s)G_1(s)U(s)$$

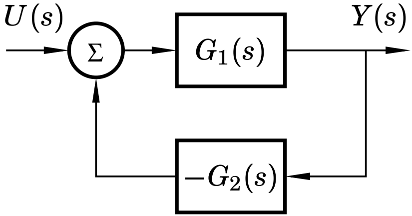
Parallel connection:



$$Y(s) = \Big(G_1(s) + G_2(s)\Big)U(s)$$

Block diagram computations with transfer functions

Feedback:

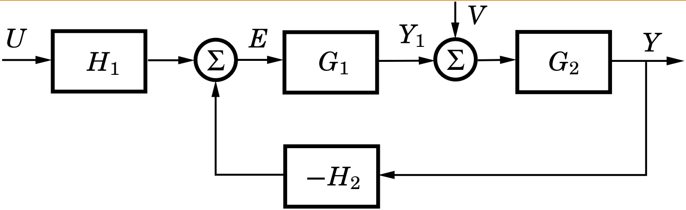


$$Y(s) = G_1(s)\Big(U(s) - G_2(s)Y(s)\Big)$$

$$Y(s)\Big(1 + G_1(s)G_2(s)\Big) = G_1(s)U(s)$$

$$Y(s) = \frac{G_1(s)}{1 + G_1G_2(s)}U(s)$$

Example



Compute transfer function from U and V to Y.

$$Y = G_2(V + Y_1)$$
$$Y_1 = G_1 E$$
$$E = H_1 U - H_2 Y$$

Solve for Y:

$$Y = \frac{G_2 G_1 H_1}{1 + G_2 G_1 H_2} U + \frac{G_2}{1 + G_2 G_1 H_2} V$$