Process control – FY

Multivariable control

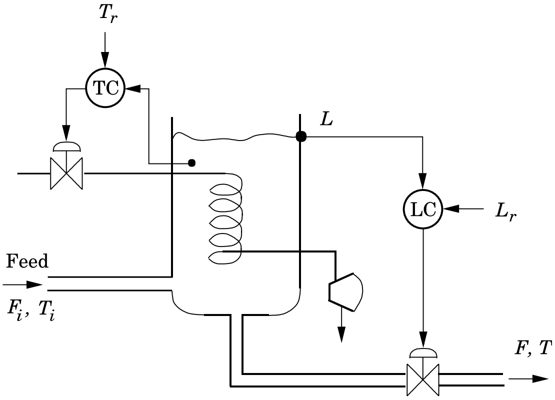
- Several inputs and outputs
- Stability and interaction
- Connect inputs and outputs (RGA)
- Eliminate interaction (Feedback)
- Model predictive control (MPC)

Reading: *Systems Engineering and Process Control:* Y.1–Y.4

Example 1: Shower flow and temperature

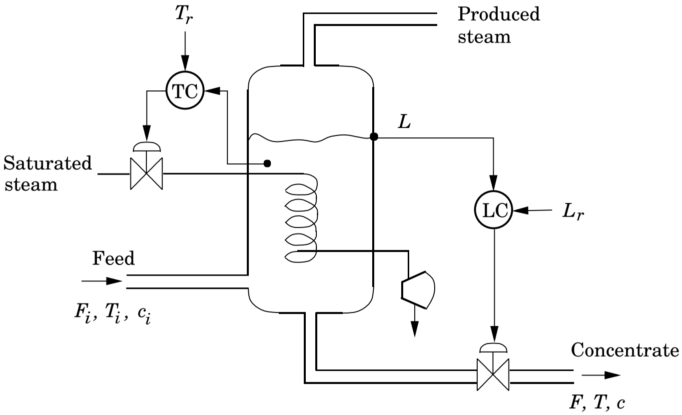
- Control signals: Cold water flow, Hot water flow
- Measurements: Total water flow, temperature
- Both inputs affect both outputs

Example 2: Tank level and temperature



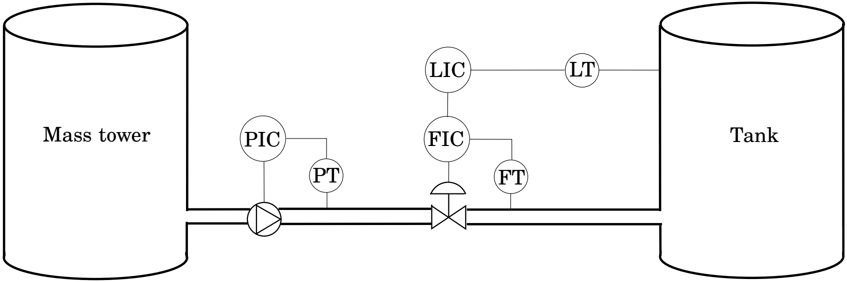
- Level control affects temperature
- Temperature control does not affect level

Example 3: Evaporator level and temperature



- Level control affects temperature
- Temperature control affects level

Example 4: Pressure and flow in transportation pipe



Pipe pressure to be kept constant, while desired flow decided by tank level controller

- Pressure control affects flow
- Flow control affects pressure

Multivariable system descriptions

State-space form:

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

where u and y are vectors

Transfer function matrix, e.g.,:

$$Y(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} U(s) = G(s)U(s)$$

Connection: $G(s) = C(sI - A)^{-1}B + D$

Multivariable system descriptions

Example: The shower

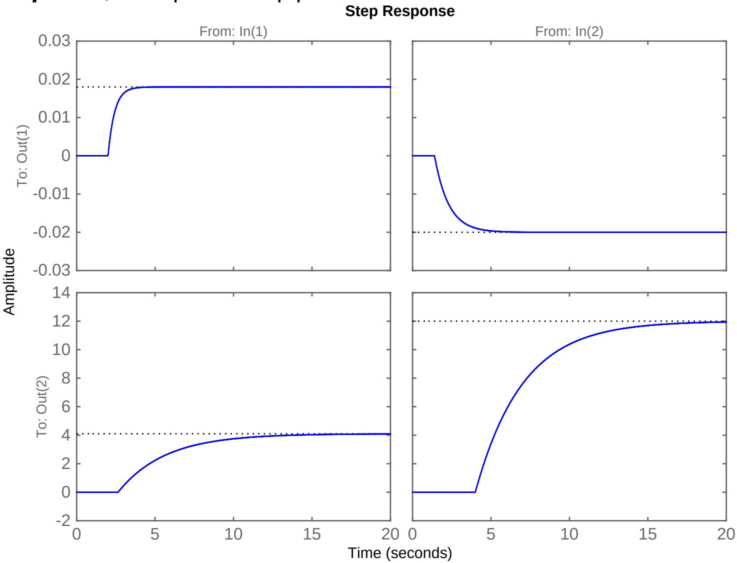
$$\begin{bmatrix} y_{\text{flow}} \\ y_{\text{temp}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -e^{-s} & e^{-s} \end{bmatrix} \begin{bmatrix} u_{\text{cold}} \\ u_{\text{hot}} \end{bmatrix}$$

Example: Transportation pipe

$$\begin{bmatrix} y_{\text{pres}} \\ y_{\text{flow}} \end{bmatrix} = \begin{bmatrix} \dfrac{0.018e^{-2s}}{0.42s+1} & \dfrac{-0.02e^{-1.4s}}{0.9s+1} \\ \dfrac{4.1e^{-2.64s}}{3s+1} & \dfrac{12e^{-4s}}{3s+1} \end{bmatrix} \begin{bmatrix} u_{\text{pump}} \\ u_{\text{valve}} \end{bmatrix}$$

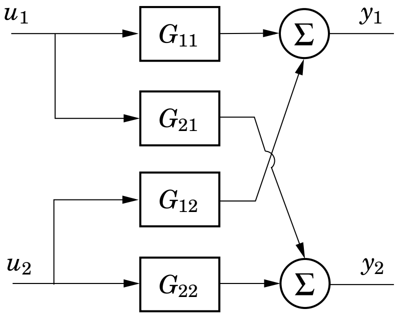
Multivariable system descriptions

Step response, transportation pipe



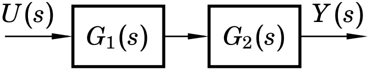
Multivariable system descriptions

Block diagrams, e.g.,:



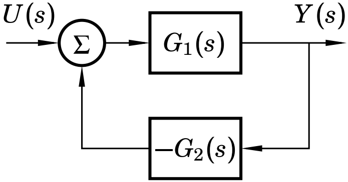
Block diagram computations for multivariable systems

Serial connection:



$$Y(s) = G_2(s)G_1(s)U(s)$$

Feedback:



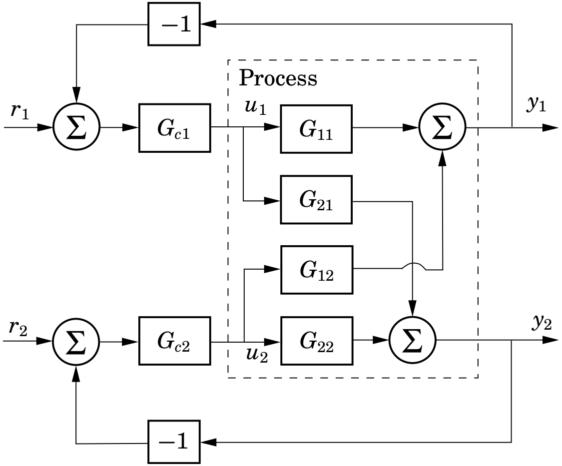
$$\begin{aligned} Y(s) &= G_1(s)\big(U(s) - G_2(s)Y(s)\big) \\ \big(I + G_1(s)G_2(s)\big)Y(s) &= G_1(s)U(s) \\ Y(s) &= \big(I + G_1(s)G_2(s)\big)^{-1}G_1(s)U(s) \end{aligned}$$

(Transfer function matrices, so order of multiplication important!)

Stability for multivariable systems

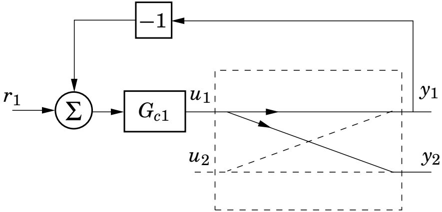
Each feedback loop by itself might be stable when no other loops closed, but full system might become unstable when more (stable) loops are closed

Example:



Stability for multivariable systems

Only first loop closed:



Output dependence on reference r_1:

$$Y_1 = \frac{G_{11}G_{c1}}{1 + G_{11}G_{c1}} R_1$$

$$Y_2 = G_{21}U_1 = \frac{G_{21}G_{c1}}{1 + G_{11}G_{c1}} R_1$$

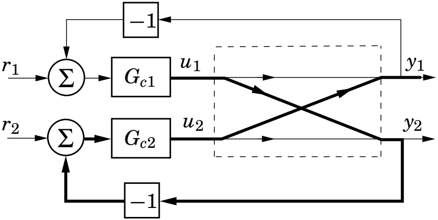
Stability decided by characteristic equation

$$1 + G_{11}G_{c1} = 0$$

(Due to symmetry, similar connection for the other loop)

Stability for multivariable systems

Both loops closed at the same time:



Output dependence on references r_1 and r_2:

$$Y_1 = \frac{G_{11}G_{c1} + G_{c1}G_{c2}(G_{11}G_{22} - G_{12}G_{21})}{A} R_1 + \frac{G_{12}G_{c2}}{A} R_2$$

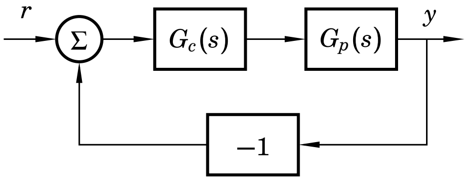
$$Y_2 = \frac{G_{21}G_{c1}}{A} R_1 + \frac{G_{22}G_{c2} + G_{c1}G_{c2}(G_{11}G_{22} - G_{12}G_{21})}{A} R_2$$

Stability decided by characteristic equation

$$A(s) = (1 + G_{11}G_{c1})(1 + G_{22}G_{c2}) - \underline{G_{12}G_{21}G_{c1}G_{c2}} = 0$$

Multivariable stability – General case

Multivariable process $G_p(s)$ and multivariable controller $G_c(s)$:



$$Y(s) = (I + G_p(s)G_c(s))^{-1} G_p(s)G_c(s)R(s)$$

Characteristic equation:

$$A(s) = \det (I + G_p(s)G_c(s)) = 0$$

Closed loop asymptotically stable $\Longleftrightarrow$ all roots in left half plane

Multivariable stability – Example

$$G_p(s) = \begin{pmatrix} \frac{1}{0.1s+1} & \frac{5}{s+1} \\ \frac{1}{0.5s+1} & \frac{2}{0.4s+1} \end{pmatrix}, \quad G_r(s) = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}$$

Each individually closed loop is stable for all $K_1, K_2 > 0$

$$\det\left(I + G_p(s)G_r(s)\right) = \det \begin{pmatrix} \frac{K_1}{0.1s+1} + 1 & \frac{5K_2}{s+1} \\ \frac{K_1}{0.5s+1} & \frac{2K_2}{0.4s+1} + 1 \end{pmatrix} = \frac{a(b)}{b(s)}$$

where

$$\begin{aligned} a(s) = 0.02s^4 &+ 0.1(3.1 + 2K_1 + K_2)s^3 \\ &+ (1.29 + 1.1K_1 + 1.3K_2 + 0.8K_1K_2)s^2 \\ &+ (2 + 1.9K_1 + 3.2K_2 + 0.5K_1K_2)s \\ &+ (1 + K_1 + 2K_2 - 3K_1K_2) = 0 \end{aligned}$$

Last coefficient negative if $3K_1K_2 > 1 + K_1 + K_2 \Rightarrow$ unstable

Relative Gain Array (RGA)

- A way to deduce coupling in “quadratic systems” (as many inputs as outputs)
- Often based on static gain $K = G(0)$
- Normalization to avoid scaling problems
- Guidance in connection of inputs and outputs

Relative gain

1. Open loop static gain when $\Delta u_k = 0, \quad k \neq j$

$$k_{ij} = G_{ij}(0) = \frac{\Delta y_i}{\Delta u_j}$$

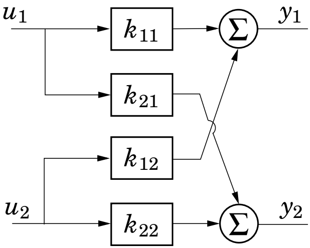
2. Closed loop static gain when $\Delta y_k = 0, \quad k \neq i$ (assume "perfect" control of the other outputs)

$$l_{ij} = \frac{\Delta y_i}{\Delta u_j}$$

3. Relative gain

$$\lambda_{ij} = \frac{k_{ij}}{l_{ij}}$$

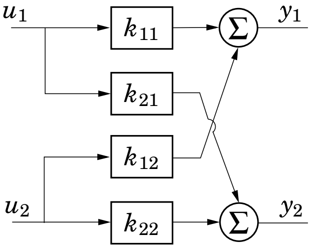
RGA for 2×2-system



Open loop gain from u_1 to y_1:

$$y_1 = k_{11} u_1$$

RGA for 2×2-system



Closed loop gain from u_1 to y_1 when all outputs controlled to zero except y_1:

$$y_2 = k_{21}u_1 + k_{22}u_2 = 0 \quad \text{(perfect control)}$$

$$u_2 = -\frac{k_{21}}{k_{22}}u_1$$

$$y_1 = \underbrace{\left(k_{11} - \frac{k_{12}k_{21}}{k_{22}}\right)}_{=l_{11}} u_1$$

RGA for 2×2-system

Relative gain:

$$\lambda_{11} = \frac{k_{11}}{l_{11}} = \frac{k_{11}}{k_{11} - \frac{k_{12}k_{21}}{k_{22}}}$$

Note that $l_{11} = \frac{\det K}{k_{22}}$

Can show that matrix with elements $1/l_{ij}$ is given by

$$\frac{1}{\det K} \begin{bmatrix} k_{22} & -k_{21} \\ -k_{12} & k_{11} \end{bmatrix} = \left(K^{-1}\right)^T$$

RGA for general quadratic system

RGA-matrix Λ with elements $\lambda_{ij} = k_{ij}\frac{1}{l_{ij}}$, can be computed as

$$\Lambda = K \mathbin{.*} \left(K^{-1}\right)^T$$

Note! "$.*$" is element-wise multiplication

For Λ, we have

- Row and column sum equal

$$\sum_{i=1}^{n} \lambda_{ij} = \sum_{j=1}^{n} \lambda_{ij} = 1$$

- System easy to control if Λ close to unit matrix (perhaps after permutations)
- Negative elements imply difficult connectivity

Connecting inputs and outputs

Rule of thumb: Inputs and outputs should be connected such that relative gain is positive and as close to one as possible.

Example 1:

Λ	Connectivity	Interaction
$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$u_1 - y_1$ $u_2 - y_2$	None
$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$	$u_1 - y_2$ $u_2 - y_1$	None
$\begin{bmatrix} 0.85 & 0.15 \\ 0.15 & 0.85 \end{bmatrix}$	$u_1 - y_1$ $u_2 - y_2$	Weak
$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$	$u_1 - y_1$ $u_2 - y_2$	Difficult

Connecting inputs and outputs

Example 2: 3×3-system

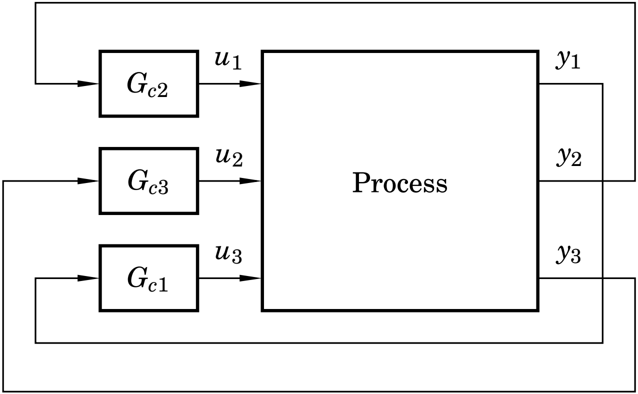
$$G(s) = \begin{pmatrix} \frac{2}{s+3} & \frac{1}{s+3} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+2} & \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+1} & \frac{2}{s+2} \end{pmatrix}$$

RGA-computation:

$$G(0) = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & 1 \\ 1 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} -2 & 0 & 3 \\ 4 & -1 & -2 \\ -1 & 2 & 0 \end{pmatrix}$$

Difficult multivariable interaction

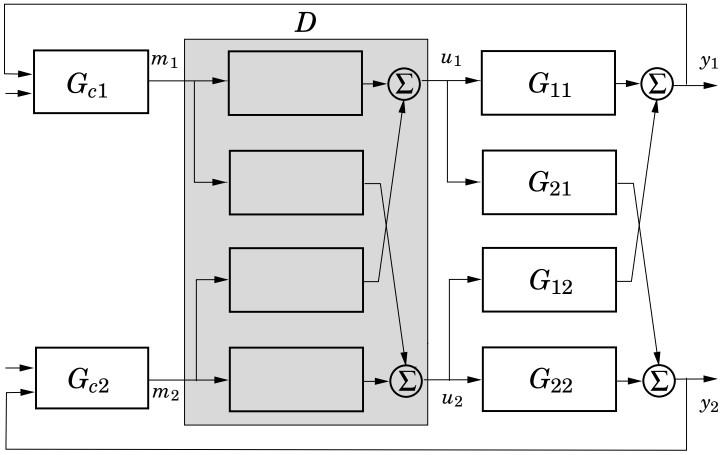
Connectivity suggestion: $u_1 - y_2, u_2 - y_3, u_3 - y_1$



G_{c2}
G_{c3}
G_{c1}
u_1
u_2
u_3
Process
y_1
y_2
y_3

Decoupling

Idea: Introduce new inputs m and decoupling filter $D(s)$ to make system easier to control



Decoupling

Outputs as a function of new inputs:

$$Y(s) = G(s)U(s) = G(s)D(s)M(s) = T(s)M(s)$$

where $T(s)$ chosen with desirable properties

- Decoupling filter: $D(s) = G(s)^{-1}T(s)$
- Choose $T(s)$ diagonal
- 2×2-case:

$$T(s) = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}, \quad D(s) = \frac{1}{\det G} \begin{bmatrix} G_{22}T_{11} & -G_{12}T_{22} \\ -G_{21}T_{11} & G_{11}T_{22} \end{bmatrix}$$

Designing the decoupling

One choice of decoupling is the following:

$$D(s) = \begin{bmatrix} 1 & -\frac{G_{12}}{G_{11}} \\ -\frac{G_{21}}{G_{22}} & 1 \end{bmatrix}$$

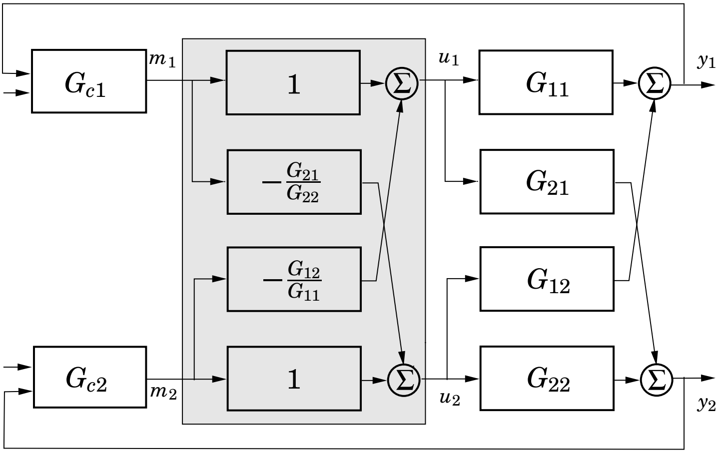
(other choices of D not covered in course)

This choice gives the following decoupled system:

$$T(s) = \begin{bmatrix} G_{11} - G_{12}\frac{G_{21}}{G_{22}} & 0 \\ 0 & G_{22} - G_{21}\frac{G_{12}}{G_{11}} \end{bmatrix}$$

This can be controlled using two ordinary controllers

Interpretation as two feedforwards:



Implementing the decoupling

Decoupling elements $-\frac{G_{12}}{G_{11}}$ and $-\frac{G_{21}}{G_{22}}$ cannot be implemented if they contain

- pure derivatives
- negative time delays

Solution options:

- low pass filter
- use static gain only

Example: Transportation pipe

Process:

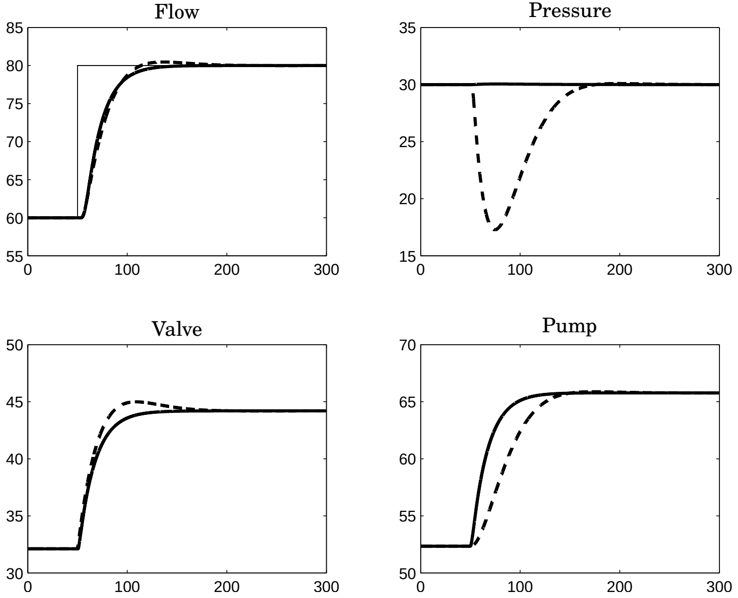
$$G(s) = \begin{bmatrix} \dfrac{0.018e^{-2s}}{0.42s+1} & \dfrac{-0.02e^{-1.4s}}{0.9s+1} \\ \dfrac{4.1e^{-2.64s}}{3s+1} & \dfrac{12e^{-4s}}{3s+1} \end{bmatrix}$$

RGA:

$$\Lambda = \begin{bmatrix} 0.73 & 0.27 \\ 0.27 & 0.73 \end{bmatrix}$$

Conclusion: Control pressure with pump and flow with valve, some interaction between loops

Simulation with decoupling (solid) and without (dashed):



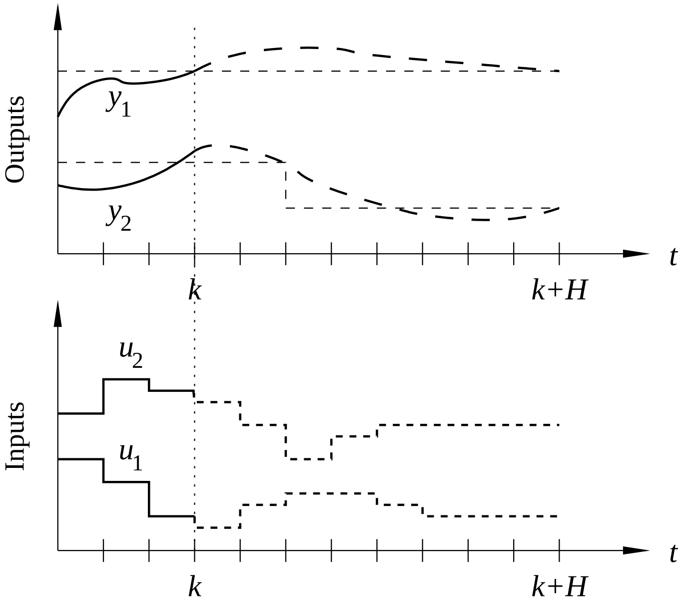
Model predictive control (MPC)

Advanced method for control of multivariable processes

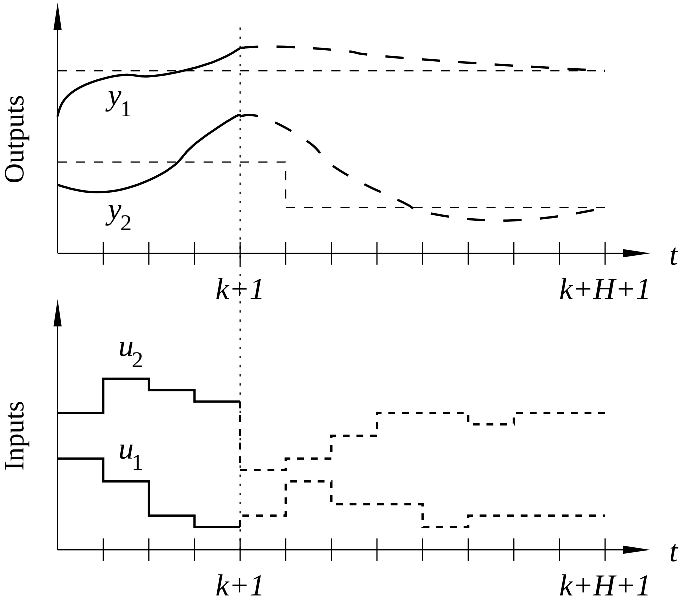
Basics:

- Need multivariable process model – linear or nonlinear
- Model predicts future behavior of system from now to time T (horizon)
- Add model, performance objective, and input and state constraints to an optimization problem
- Online solution to optimization problem in every sample
  - Gives optimal trajectories for inputs
  - Use first value of input trajectory only
  - Repeat optimization in next sample (receding horizon)

Optimization over receding horizon



Optimization over receding horizon



Optimization problem example

Linear model:

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

Quadratic objective function:

$$J = \sum_{t=k+1}^{k+H} \left(w_1(r_1 - y_1)^2 + w_2(r_2 - y_2)^2 \right) + \sum_{t=k}^{k+H-1} \left(\rho_1 \Delta u_1^2 + \rho_2 \Delta u_2^2 \right)$$

Optimization problem: Minimize J w.r.t. u_1 and u_2 while taking constraints into account:

$$u_{\mathrm{i}_{\min}} \leq u_i \leq u_{\mathrm{i}_{\max}}$$

Multivariable control – Summary

- Unidirected influence from control loop to another does not affect stability and the effect can be eliminated by feedforward
- Multivariable interaction can affect both stability and performance to the worse. Interaction can be reduced with decoupling filter
- Analysis tool RGA describes the difficulty of the interaction and what outputs that should be fed back to what inputs
- Difficult interaction or high performance demands might require a multivariable controller, e.g., MPC