Mathematical Modelling FK (FRT095)
F2

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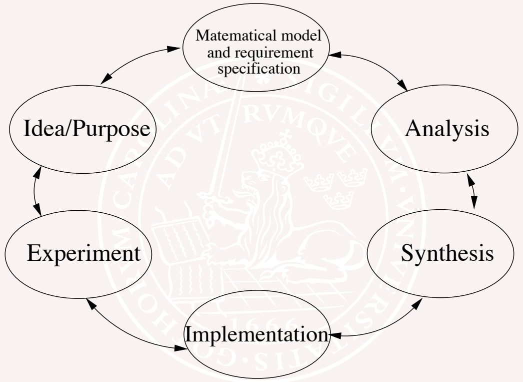
Project

- Project supervision from
  - Mathematics, Mathematical Statistics, Automatic Control.
- Project plan. An A4-paper prepared after consulting the supervisor. Send to course responsible by January 31. Use email with subject line “FRT095”.
- Written report
- Oral presentation (shared among all group members)
- Opposition (all team members together)
  Written opposition report
- 4 persons per project (possibly less)

Modelling in three phases:

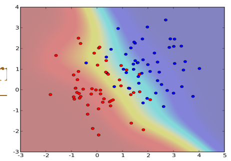
1. Problem structure
   - **Formulate purpose**, requirements for accuracy
   - Break up into subsystems — What is important?
2. Basic equations
   - Write down the relevant physical laws
   - Collect experimental data
   - Test hypotheses
   - Validate the model against fresh data
3. Model with desired features is formed
   - Put the model on suitable form.
     (Computer simulation or pedagogical insight?)
   - Document and illustrate the model
   - Evaluate the model: **Does it meet its purpose?**



Lecture 2

- Statistical modeling from data (static black boxes)
  - Singular Value Decomposition (SVD)
  - Principal Component Analysis (Factor Analysis)
- **Neural Networks** / **Machine learning**
- Dynamic experiments (dynamic black boxes)
  - Step response
  - Frequency response
  - Correlation analysis
- Gray boxes
  - Prediction error methods
  - Differential-Algebraic Equations revisited

Components for deep lear



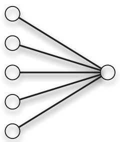
- One neuron
  - Example: Logistic regression
  - Classification model (x feature vector, (w,b) parameters, s smooth thresholding
    $x \in R^d, w \in R^d, b \in R, f(x) = s(w^T x + b)$
  - Logistic regression $\quad s(z) = \dfrac{1}{1+e^{-x}}$
  - ML estimate of parameters (w,b) is a convex optimization problem

$$\min_{w} \quad \frac{1}{2} w^T w + C \sum_{i=1}^{l} \log(1 + e^{-y_i w^T x_i}).$$

Single Layer Neural Networks
One Neuron

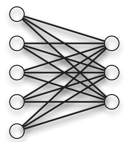
- One neuron

$x \in R^d, w \in R^d, b \in R, f(x) = s(w^T x + b)$



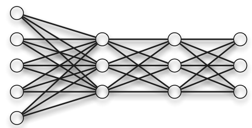
Single Layer Neural Networks
Several Neurons

- Several parallell neurons
  $x \in R^d, y \in R^k, B \in R^d, W - k \times \text{dmatrix}$
  $y = s(Wx + B)$
- Elementwise smooth thresholding – s



Deep Neural Networks
Many layers

- However
- Naively implemented would give to many parameters
- Example
- 1M pixel image
- 1M hidden layers
- 10^{12} parameters between each pairs of layers



Fun stuff before we get started

Deep Dream version

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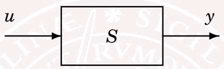
Basic idea of system identification



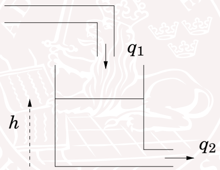
Measure U and y. Figure out a model of S, consistent with measured data.

Important aspects:

- We can only measure the u and y in discrete time points (sampling). Can be natural to use the discrete-time models.
- The system is affected by interference and measurement errors. We may need to signal models for dealing with this.

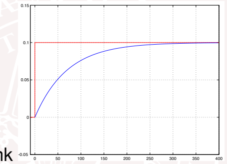
Example

A tank which attenuates flow variations in q_1. Characterization of the tank system:



- Input: q_1
- Output: q_2 and/or h
- Internal variables / conditions: h

Step response

Step response for the tank 

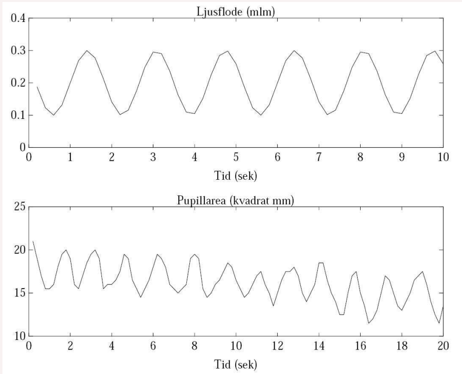
Can give idea of the dominant time constant, static reinforcement, character (overshoot or not)

Frequency response

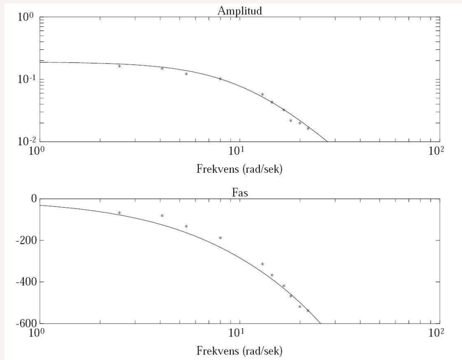
For good signal-to-noise ratio, an estimate of $G(i\omega)$ is obtained directly from the amplitudes and phase positions of u, y

$$\begin{aligned} u(t) &= A \sin \omega t \\ y(t) &= A|G(i\omega)| \sin(\omega t + \arg G(i\omega)) \end{aligned}$$

How light affects pupil area



Bode-diagram for pupil



Correlation analysis

Can we estimate the impulse response with other inputs?

- Impulse response formula in discrete time ($T = 1$, $v =$ noise):

$$y(t) = \sum_{k=1}^{\infty} g_k u(t-k) + v(t)$$

- If v white noise with $\mathbf{E}v^2 = 1$, then

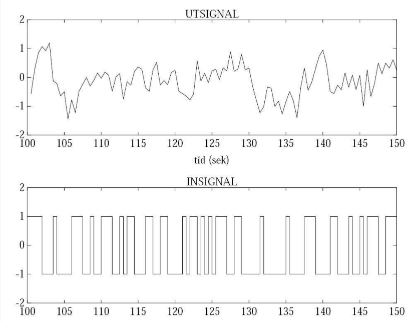
$$R_{yu}(k) = \mathbf{E}y(t)u(t-k) = g_k$$

- Covariance R_{yu} estimated by N data points with

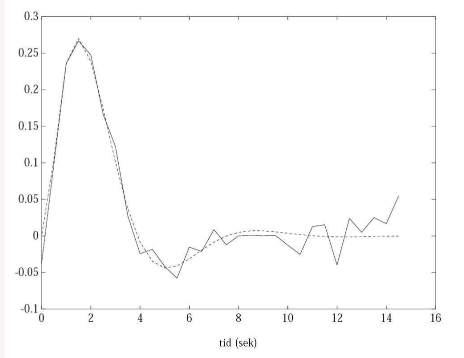
$$\widehat{R}_{yu}^N(k) = \frac{1}{N}\sum_{t=1}^{N} y(t)u(t-k)$$

Example

Correlation analysis for $\frac{1}{s^2+2s+1}$ (in- and out-put data)



Estimated and actual impulse responses



Basic rules

Make experiments with conditions similar to the conditions in which the model is to be used!

(Models from step response can be expected to work best on the stage.)

Save some data for model validation, i.e. check the model with data set different from the one that generated the model!

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Prediction Error Methods

Find the unknown parameters θ by optimization:

$$\min_{\theta} \|\widehat{y}(t,\theta) - y(t)\|$$

Here $y(t)$ is the measured output at time t and $\widehat{y}(t,\theta)$ is the predicted output based on past measurements using a model with parameter values θ.

Prediction Error Method with Repeated Simulation

For a nonlinear grey-box model

$$0 = F(\dot{x}, x, t, \theta)$$
$$y(t) = h(x, t, \theta)$$

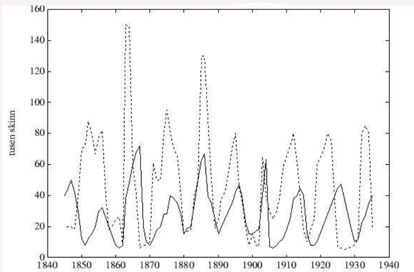
the unknown parameters θ could be determined by the prediction error method

$$\min_{\theta} \|\widehat{y}(t,\theta) - y(t)\|$$

where the output prediction $\widehat{y}(t,\theta)$ is computed by simulation.

(Repeated simulation for different values of θ could however be very time-consuming.)

Population dynamics / Ecology



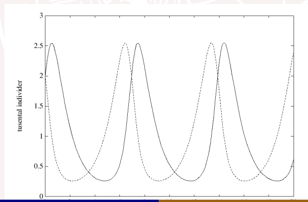
Variations in the number of lynx (solid) and hares (dashed) in Canada. Can you predict the periodic variations?

Population dynamics

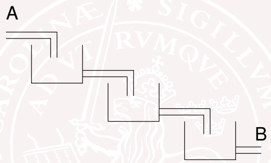
N_1 number of lynx, N_2 number of hares

$$\frac{d}{dt}N_1(t) = (\lambda_1 - \gamma_1)N_1(t) + \alpha_1 N_1(t)N_2(t)$$
$$\frac{d}{dt}N_2(t) = (\lambda_2 - \gamma_2)N_1(t) - \alpha_2 N_1(t)N_2(t)$$

Simulation:



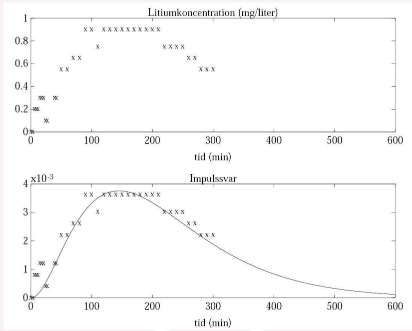
Mixing tanks in Skärblacka paper factory



A linear transfer function of three series-connected mixing tanks has the form $\frac{1}{(s\theta+1)^3}$.

To determine θ, radioactive lithium is added in **A**. Radioactivity was then measured by **B** as a function of time.

Impulse response



In the lower picture, θ has been chosen to adapt to the impulse

Grey Models — the best of both worlds

- White boxes: Physical laws provide some insight
- Black boxes: Statistics estimates complex relationships
- Gray boxes: Combine simplicity with insight