Welcome to Mathematical Modelling FK (FRT095)

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Matematisk Modellering FK (FRT095)

Course homepage:

`http://www.control.lth.se/course/FRT095`

- 4.5 högskolepoäng ; betyg U/G
- 4 h lectures
- 100 h project

Project

- Project supervision from
  - Mathematics, Mathematical Statistics, Automatic Control.
- Project plan. An A4-paper prepared after consulting the supervisor. Send to course responsible by January 31. Use email with subject line "FRT095".
- Written report
- Oral presentation (shared among all group members)
- Opposition (all team members together)
  Written opposition report
- 4 persons per project (possibly less)

Written report

See the website instructions:

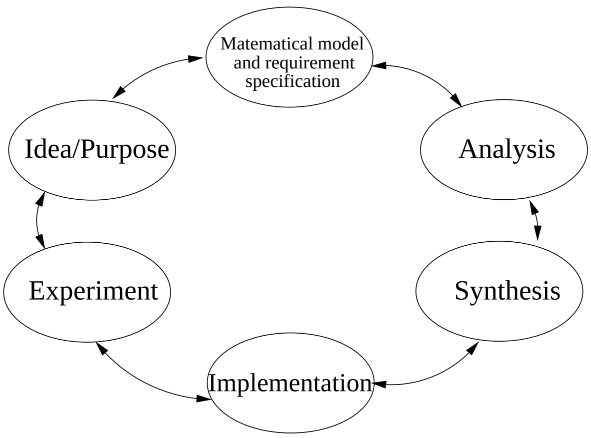
- Cover sheet
- Summary
- Table of Contents
- Main Text
  - Presentation of problem: What is the purpose of the model?
  - Summary of used literature
  - Theory/Method
  - Implementation
  - Results
  - Evaluation/discussion: Does the model suit its purpose?
  - Reference list
- Description of how the work is distributed within the group
- Presentation of the course theory part

Mathematical modelling — Why and How?

- Why modelling?
  - Natural sciences: Models for analysis (understanding)
  - Engineering sciences: Models for synthesis (design)
  - Specification: Model of a good technical solution
- White boxes: Physical modeling
  Model derived from fundamental physical laws
- Black boxes: Statistical methods and machine learning
  Model derived from measurement data
  - Singular Value Decomposition (SVD)
  - Principal Component Analysis (Factor Analysis)
  - Neural Networks / Machine learning
  - System Identification / Time Series Analysis
- Gray boxes: Combination of the two

All Three Modelling Phases Must be Described

1. Problem structure
   - **Formulate purpose**, requirements for accuracy
   - Break up into subsystems — What is important?
2. Basic equations
   - Write down the relevant physical laws
   - Collect experimental data
   - Test hypotheses
   - Validate the model against fresh data
3. Model with desired features is formed
   - Put the model on suitable form.
     (Computer simulation or pedagogical insight?)
   - Document and illustrate the model
   - Evaluate the model: **Does it meet its purpose?**



Engineering Ethics [1]

- Relevant for the Pi-program?
- Ethical linear algebra?
- Ethical mathematical modelling?

[1] Thanks to Maria Henningsson Pi-02 for suggesting the next few slides.

“Our calculations show that...”

- What is behind the numbers?
- What assumptions are made?
- What limitations are there?

"Essentially, all models are wrong, but some are useful."
- George E. P. Box.

Knowledge Gives You Power and Responsibility

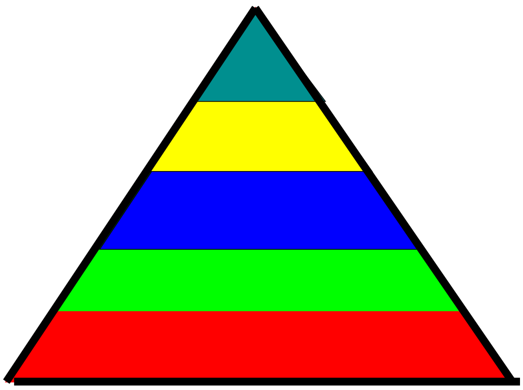
- Your expert role will give you an advantage
- What assumptions are made?
- What limitations are there?

Example1: The CitiCorp Building

Example 2: The Parental Leave Insurance

What percentage of your income do you get?

Example 3: Mortage Securities



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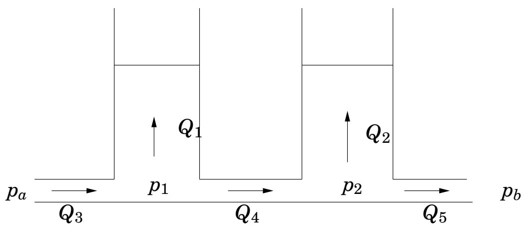
White boxes

Physical modeling (white boxes)

- **Analogies between different fields**
- Dimensioned and dimensionless variables
- Subsystems and differential-algebraic equations

Principles and analogies: Hydraulics

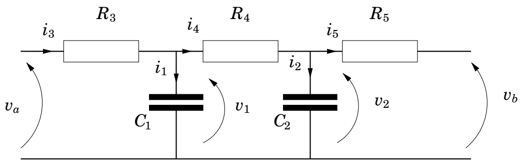
Example 1. A hydraulic system:



Incompressible fluid. Pressures: p_a, p_1, p_2, and p_3.
Volume flows: Q_1, Q_2, Q_3, Q_4, and Q_5.

Principles and analogies: Electrics

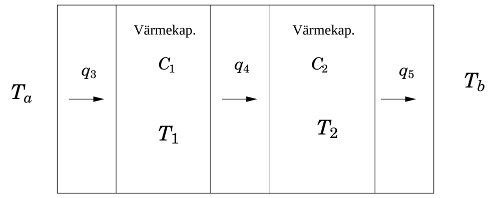
Example 2. An electrical system:



Potentials v_a , v_b , v_1, and v_2
Currents i_1 , i_2, i_3, i_4, and i_5

Principles and analogies: Heat

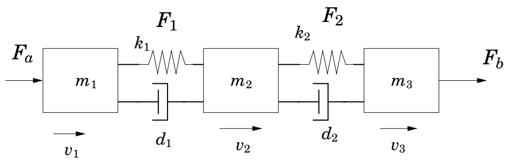
Example 3. A thermal system
(heat transfer through a wall):



Two elements with thermal capacities C_1 and C_2 separated by insulating layers. Heat flows: q_3 , q_4 and q_4
Temperatures: T_a , T_b , T_1 and T_2

Principles and analogies: Mechanics

Exempel 4. A mechanical system:



External forces: F_a and F_b
Velocities: v_1, v_2 and v_3
Spring constants: k_1 and k_2
Damping constants: d_1 and d_2

Analogies

Analogies: hydraulic - electric - thermal - mechanical
Two types of variables:

A. Flow Variables
- volume flow
- power flow
- heat flow
- speed

B. Intensity variables
- pressure
- voltage
- temperature
- force

For both of them addition rules hold.

Analogies (cont'd)

Intensity variations

$$C \cdot \frac{d}{dt}(\text{intensity}) = \text{flow}$$

C "capacitance":
hydraulic: $A/(\rho g)$
electrical: kapacitans
heat: thermal capacity
mechanical: inverse spring constant
Balance equations!

(More complicated if the capacitance is not constant.)

Analogies (cont'd)

Losses

$$\text{flow} = \phi(\text{intensity})$$

$$\text{intensity} = \varphi(\text{flow})$$

Hydraulic: flow resistance
Electrics: resistance
Heat: thermal conductivity
Mechanics: friction

Often linear relationship in the electrical case - nonlinearly in the other (may be approximated by linear for small changes of variables)

More phenomena

Intensity variations

$$L \cdot \frac{d}{dt}(\text{flow}) = \text{intensity}$$

L "inductance"
hydraucs: $\rho l/A$
electrics: inductans
heat: –
mechanics: mass
balance equations!

(more complicated if the inductance is not constant.)

Energy flows

Can you make a general modeling theory based on flow and intensity variables? Note the following.

$$\begin{aligned} \text{pressure} \cdot \text{flow} &= \text{power} \\ \text{voltage difference} \cdot \text{current} &= \text{power} \\ \text{force} \cdot \text{velocity} &= \text{power} \\ \text{torque} \cdot \text{angular velocity} &= \text{power} \\ \text{temperature} \cdot \text{heat flow} &= \text{power} \cdot \text{temperature} \end{aligned}$$

White boxes

Physical modeling (white boxes)

- Analogies between different fields
- **Dimensioned and dimensionless variables**
- Subsystems and differential-algebraic equations

Dimension analysis

Physical variables have dimensions. E.g.,

$$[\text{density}] = ML^{-3}$$
$$[\text{force}] = M \cdot \frac{L}{T^2} = MLT^{-2}$$

where

$$M = [\text{mass}], \quad T = [\text{time}], \quad L = [\text{length}]$$

Physical connections must be dimensionally "correct".

Example: Bernoulli's law

In Bernoulli's law $v = \sqrt{2gh}$ you have

$$[v/\sqrt{gh}] = LT^{-1}(LT^{-2}L)^{-0.5} = 1$$

$v/\sqrt{gh}$ is an example of dimensionless quantity.

Dimensionless quantities and scaling

Some historical passanger ships:

- Kaiser Wilhelm the great, 1898, 22 knots, 200 m
- Lusitania, 1909, 25 knots, 240 m
- Rex, 1933, 27 knots, 269 m
- Queen Mary, 1938, 29 knots, 311 m

Note that the ratio *(velocity)*^{2}*/(length)* is almost constant

Which physical phenomenon can be thought to be the cause?

2 min problem

Find the relationship (except for a scaling by a dimensionless constant) between a pendulum period time and its mass, its length and the acceleration of gravity g, i.e.,

$$t = f(m, l, g)$$

White boxes

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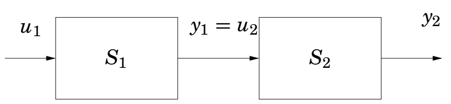
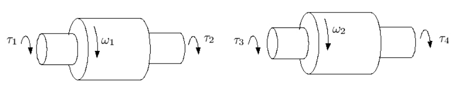
Block models

Boxes linked by identifying the output of one with input of another. Series connection of two state models gives a new state model.



Block models (*à la* Simulink) often requires a predetermined causality which can bee problematic. We want to be able to model in general without first determining what is input and what is output.

Examples of more general connection:



State models for two separate components:

$$\dot{\phi}_1 = \omega_1 \qquad \dot{\phi}_2 = \omega_2$$
$$J_1\dot{\omega}_1 = \tau_1 + \tau_2 \qquad J_2\dot{\omega}_2 = \tau_3 + \tau_4$$

Connection:

$$\phi_1 = \phi_2$$
$$\tau_2 = -\tau_3$$

The resulting model is not exactly a state model.

Linear differential-algebraic equations (DAE)

$$E\dot{z} = Fz + Gu$$

If E were non-singular, one could write

$$\dot{z} = E^{-1}Fz + E^{-1}Gu$$

which is a valid state model. If E is singular, variables have to be eliminated to get a state equation. Using a DAE solver is often better, since elimination can destroy sparsity.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & J_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\omega}_1 \\ \dot{\phi}_2 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix}$$

Nonlinear differential-algebraic equations (DAE)

Differential-algebraic equations, DAE

$$F(\dot{z}, z, u) = 0, \quad y = H(z, u)$$

u: input, y: output, z: "internal variable"

Special case: state model

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

u: input, y: output, x: state

Example: Pendulum

A pendulum with length L and position coordinates (x, y) moves according to the equations

$$\dot{x} = u \qquad \dot{y} = v$$
$$\dot{u} = \lambda x \qquad \dot{v} = \lambda y - g \qquad L^2 = x^2 + y^2$$

Differentiating the fifth equation gives

$$0 = \dot{x}x + \dot{y}y = ux + vy$$

Differentiating a second time gives

$$\begin{aligned} 0 &= \dot{u}x + u\dot{x} + \dot{v}y + v\dot{y} \\ &= \lambda x^2 + u^2 + (\lambda y - g)y + v^2 \\ &= \lambda L^2 - gy + u^2 + v^2 \end{aligned}$$

and a third time

$$0 = \dot{\lambda}L^2 - 3gv$$

Finally, we have derivative expressions for all variables!

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Black Boxes

- **Statistical modeling from data (static black boxes)**
  - Singular Value Decomposition (SVD)
  - Principal Component Analysis (Factor Analysis)
- Neural Networks / Machine learning
- Dynamic experiments (dynamic black boxes)
  - Step response
  - Frequency response
  - Correlation analysis
- Gray boxes
  - Prediction error methods
  - Differential-Algebraic Equations revisited

Singular Value Decomposition (SVD)

A matrix M can always be factorized

$$M = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*$$

with Σ diagonal and invertible and U, V unitary:

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \qquad U^*U = I \qquad V^*V = I$$

Diagonal elements of Σ are called singular values of M and correspond to the square roots of the eigenvalues of M^*M.

Computation of SVD is *very numerically stable*.

Example of SVD

$$M \quad = \quad U \quad \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \quad V^*$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{U} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{\sqrt{2}}}_{V^*}$$

What does it mean if a singular value is zero?

What does it mean if it is near zero?

Good children can have many names

Collect all the data into a large matrix. Then compute the **SVD**:

$$\begin{bmatrix} y_1(1) & y_1(2) & \dots & y_1(N) \\ y_2(1) & y_2(2) & \dots & y_2(N) \\ \vdots & & & \\ y_p(1) & y_p(2) & \dots & y_p(N) \end{bmatrix} = U \underbrace{\begin{bmatrix} \sigma & & 0 \\ & \ddots & \\ 0 & & \sigma_p \end{bmatrix}}_{\Sigma} V^*$$

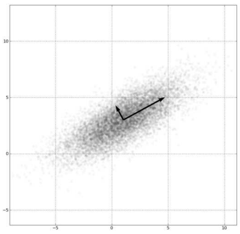
Singular values σ_i in decreasing order on the diagonal of Σ. The first columns of U give the direction of the main data area.

Principal Component Analysis: By replacing the small singular values σ_i with zeros focuses on the essential.

The name '**factor analysis**' is sometimes used as a synonymous, since large singular values σ_i highlight important factors.

Principal Component Analysis (PCA)

Data from a bi-dimensional Gaussian distribution centered in $(1, 3)$:



Principal component $(0.878, 0.478)$ has standard deviation 3.

Next component has standard deviation 1.

[Källa: Wikipedia]

Example: Image processing

What does this picture represent?

```
M =

     1     0     0     1     1     1     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     1     1
     1     0     0     0     1     0     0     1     0     1
     1     1     0     0     1     0     0     1     0     1
```

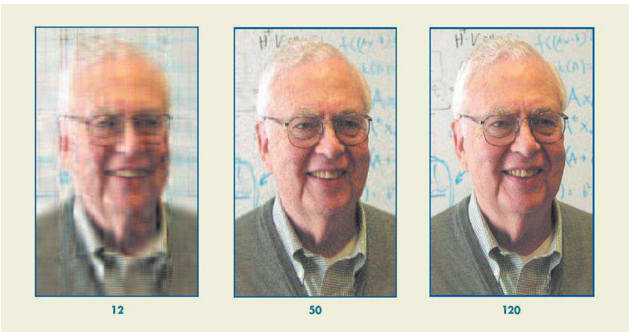
Example: Image processing with SVD

```
>> [U,S,V]=svd(M)

U =

   -0.4747    0.8662    0.0000   -0.1559    0.0000
   -0.4291   -0.1371   -0.0000    0.5450   -0.7071
   -0.4508   -0.3256   -0.7071   -0.4368   -0.0000
   -0.4291   -0.1371   -0.0000    0.5450    0.7071
   -0.4508   -0.3256    0.7071   -0.4368    0.0000

S =

4.5638 0      0      0      0      0      0      0      0      0
0      1.3141 0      0      0      0      0      0      0      0
0      0      1.0000 0      0      0      0      0      0      0
0      0      0      0.6670 0      0      0      0      0      0
0      0      0      0      0.0000 0      0      0      0      0
```

Example: Image processing with SVD

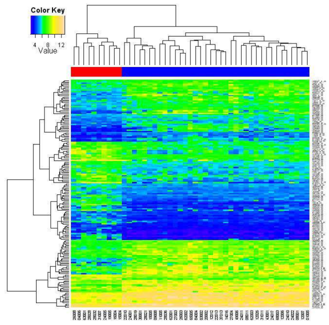
```
round(U*S1*V') =

     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1

round(U*S2*V') =

     1     0     0     1     1     1     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
     1     0     0     0     1     0     0     1     0     1
```

Example: Image processing



The original image has 897-by-598 pixels. Tacking red, green and blue vertically gives a 2691-by-598 matrix. Truncating all but 12 singular values gives the left picture. 120 gives the right.

Example: Correlations genes-proteines



Cancer research: microarrays (glass) with human genes are exposed to healthy cells, then to sick ones. Make a SVD of the data to find out which genes are important!

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Tomorrow

- More on black and grey models
- Registered students will get a project
- Karl Johan Åström will lecture on bicycle modelling!