# Welcome to Mathematical Modelling FK (FRT095)

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# Matematisk Modellering FK (FRT095)

Course homepage:

`http://www.control.lth.se/course/FRT095`

- 4.5 credits (grade Pass or Fail)
- 4 h lectures (21/1-16 and 22/1-16)
- 100 h project

# Project

- Project supervision from
  - Mathematics, Mathematical Statistics, Automatic Control.
- Project plan. An A4-paper prepared after consulting the supervisor. Sent to the course responsible by email by 5/2-16.
- Written report
- Oral presentation (shared among all group members)
- Opposition (all team members together)
  Written opposition report
- 4 persons per project

# Mathematical modelling — Lectures

- Why modelling?
  - Natural sciences: Models for analysis (understanding)
  - Engineering sciences: Models for synthesis (design)
  - Specification: Model of a good technical solution

- Physical modeling (white boxes, **today**)
  Model derived from fundamental physical laws

- Statistical methods (black boxes, **today**)
  Model derived from measurement data
  - Singular Value Decomposition (SVD)
  - Principal Component Analysis (Factor Analysis)
  - System Identification / Time Series Analysis

- Combination of the two (gray boxes, **today or tomorrow**)

- Bikes and Projects (**tomorrow**)

# Engineering Ethics [1]

- Relevant for the Pi-program?
- Ethical linear algebra?
- Ethical mathematical modelling?

[1] Thanks to Maria Henningsson Pi-02 for suggesting the next few slides.

# “Our calculations show that...”

- What is behind the numbers?
- What assumptions are made?
- What limitations are there?

”Essentially, all models are wrong, but some are useful.”
- George E. P. Box.

# Knowledge Gives You Power and Responsibility

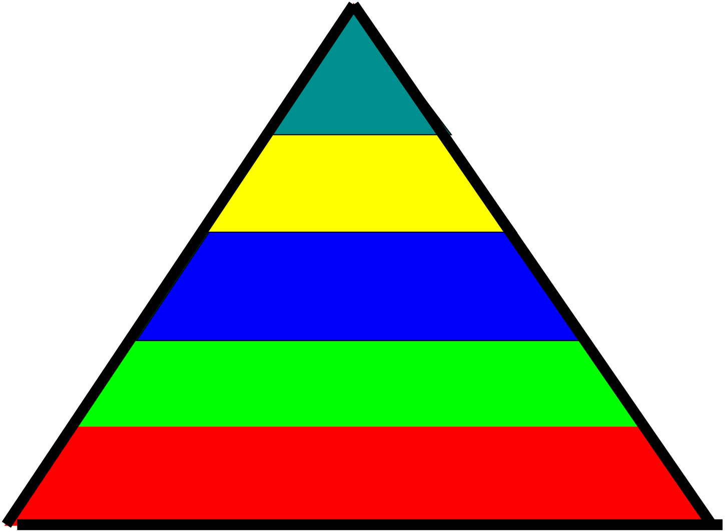
- Your expert role will give you an advantage
- What assumptions are made?
- What limitations are there?

# Example1: The CitiCorp Building

# Example 2: The Parental Leave Insurance

What percentage of your income do you get?

# Example 3: Mortgage Securities

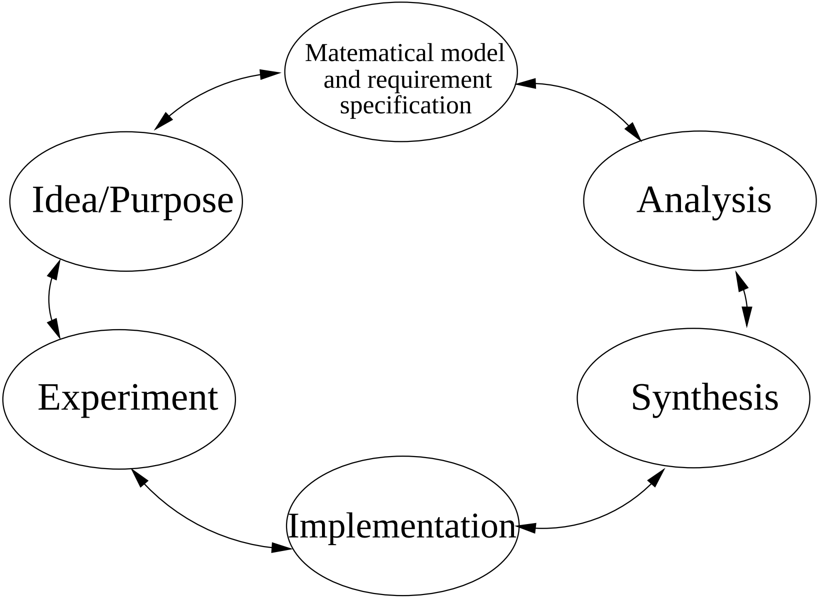


# Modelling in three phases:

1. Problem structure
   - **Formulate purpose**, requirements for accuracy
   - Break up into subsystems — What is important?

2. Basic equations
   - Write down the relevant physical laws
   - Collect experimental data
   - Test hypotheses
   - Validate the model against fresh data

3. Model with desired features is formed
   - Put the model on suitable form.
     (Computer simulation or pedagogical insight? )
   - Document and illustrate the model
   - Evaluate the model: **Does it meet its purpose?**


Matematical model
and requirement
specification
Analysis
Synthesis
Implementation
Experiment
Idea/Purpose

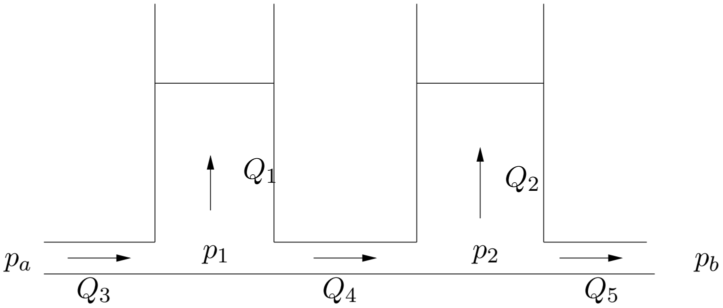
# Physical Modeling

Physical modeling (white boxes)

- **Analogies between different fields**
- Dimensioned and dimensionless variables
- Subsystems and differential-algebraic equations

# Principles and analogies: Hydraulics

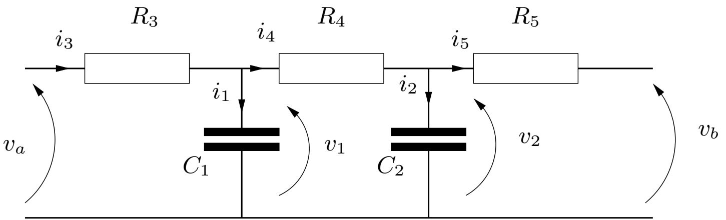
Example 1. A hydraulic system:




Incompressible fluid. Pressures: $p_a$, $p_1$, $p_2$, and $p_3$.
Volume flows: $Q_1$, $Q_2$, $Q_3$, $Q_4$, and $Q_5$.

# Principles and analogies: Electrics

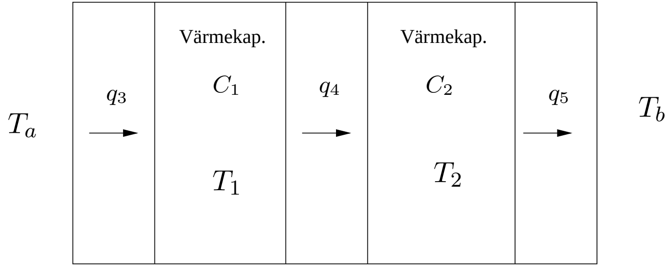
Example 2. An electrical system:




Potentials $v_a$ , $v_b$ , $v_1$, and $v_2$
Currents $i_1$ , $i_2$, $i_3$, $i_4$, and $i_5$

# Principles and analogies: Heat

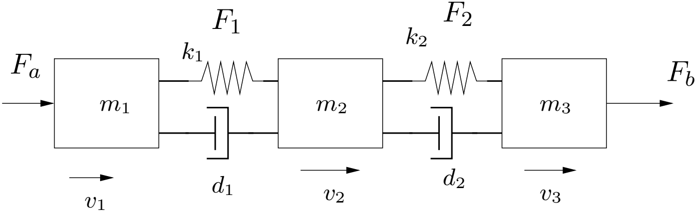
Example 3. A thermal system
(heat transfer through a wall):




Two elements with thermal capacities $C_1$ and $C_2$ separated by insulating layers. Heat flows: $q_3$ , $q_4$ and $q_4$
Temperatures: $T_a$ , $T_b$ , $T_1$ and $T_2$

# Principles and analogies: Mechanics

Exempel 4. A mechanical system:




External forces: $F_a$ and $F_b$
Velocities: $v_1$, $v_2$ and $v_3$
Spring constants: $k_1$ and $k_2$
Damping constants: $d_1$ and $d_2$

# Analogies

Analogies: hydraulic - electric - thermal - mechanical
Two types of variables:

A. Flow Variables
   - volume flow
   - power flow
   - heat flow
   - speed
B. Intensity variables
   - pressure
   - voltage
   - temperature
   - force

For both of them addition rules hold.

# Analogies (cont'd)

Intensity variations

$$C \cdot \frac{d}{dt}(\text{intensity}) = \text{flow}$$

$C$ "capacitance":
hydraulic: $A/(\rho g)$
electrical: kapacitans
heat: thermal capacity
mechanical: inverse spring constant
Balance equations!
(More complicated if the capacitance is not constant.)

# Analogies (cont'd)

Losses

$$\text{flow} = \phi(\text{intensity})$$

$$\text{intensity} = \varphi(\text{flow})$$

Hydraulic: flow resistance
Electrics: resistance
Heat: thermal conductivity
Mechanics: friction

Often linear relationship in the electrical case - nonlinearly in the other (may be approximated by linear for small changes of variables)

# More phenomena

Intensity variations

$$L \cdot \frac{d}{dt}(\text{flow}) = \text{intensity}$$

$L$ "inductance"
hydraucs: $\rho l / A$
electrics: inductans
heat: –
mechanics: mass
balance equations!

(more complicated if the inductance is not constant.)

# Energy flows

Can you make a general modeling theory based on flow and intensity variables? Note the following.

$$
\begin{aligned}
\text{pressure} \cdot \text{flow} &= \text{power} \\
\text{voltage difference} \cdot \text{current} &= \text{power} \\
\text{force} \cdot \text{velocity} &= \text{power} \\
\text{torque} \cdot \text{angular velocity} &= \text{power} \\
\text{temperature} \cdot \text{heat flow} &= \text{power} \cdot \text{temperature}
\end{aligned}
$$

# Physical modeling

Physical modeling (white boxes)

- Analogies between different fields
- **Dimensioned and dimensionless variables**
- Subsystems and differential-algebraic equations

# Dimension analysis

Physical variables have dimensions. E.g.,

$$[\text{density}] = ML^{-3}$$

$$[\text{force}] = M \cdot \frac{L}{T^2} = MLT^{-2}$$

where

$$M = [\text{mass}], \quad T = [\text{time}], \quad L = [\text{length}]$$

Physical connections must be dimensionally “correct”.

# Example: Bernoulli's law

In Bernoulli's law $v = \sqrt{2gh}$ you have

$$[v/\sqrt{gh}] = LT^{-1}(LT^{-2}L)^{-0.5} = 1$$

$v/\sqrt{gh}$ is an example of dimensionless quantity.

# Dimensionless quantities and scaling

Some historical passanger ships:

- Kaiser Wilhelm the great, 1898, 22 knots, 200 m
- Lusitania, 1909, 25 knots, 240 m
- Rex, 1933, 27 knots, 269 m
- Queen Mary, 1938, 29 knots, 311 m

Note that the ratio $\textit{(velocity)}^2/\textit{(length)}$ is almost constant
Which physical phenomenon can be thought to be the cause?

# 2 min problem

Find the relationship (except for a scaling by a dimensionless constant) between a pendulum period time and its mass, its length and the acceleration of gravity $g$, i.e.,

$$t = f(m, l, g)$$

# Lecture 1

Physical modeling (white boxes)

- Analogies between different fields
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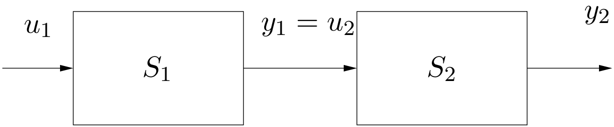
- **Subsystems and differential-algebraic equations**

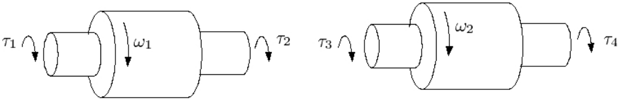
# Block models

Boxes linked by identifying the output of one with input of another. Series connection of two state models gives a new state model.




Block models (*à la* Simulink) often requires a predetermined causality which can bee problematic. We want to be able to model in general without first determining what is input and what is output.

# Examples of more general connection:




State models for two separate components:

$$\dot{\phi}_1 = \omega_1 \qquad \dot{\phi}_2 = \omega_2$$

$$J_1\dot{\omega}_1 = \tau_1 + \tau_2 \qquad J_2\dot{\omega}_2 = \tau_3 + \tau_4$$

Connection:

$$\phi_1 = \phi_2$$

$$\tau_2 = -\tau_3$$

The resulting model is not exactly a state model.

# Linear differential-algebraic equations (DAE)

$$E\dot{z} = Fz + Gu$$

If $E$ were non-singular, one could write

$$\dot{z} = E^{-1}Fz + E^{-1}Gu$$

which is a valid state model. If $E$ is singular, variables have to be eliminated to get a state equation. Using a DAE solver is often better, since elimination can destroy sparsity.

**Example:**

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & J_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\omega}_1 \\ \dot{\phi}_2 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \omega_1 \\ \phi_2 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ \tau_4 \end{bmatrix}$$

# Nonlinear differential-algebraic equations (DAE)

Differential-algebraic equations, DAE

$$F(\dot{z}, z, u) = 0, \quad y = H(z, u)$$

$u$: input, $y$: output, $z$: "internal variable"

Special case: state model

$$\dot{x} = f(x, u), \quad y = h(x, u)$$

$u$: input, $y$: output, $x$: state

# Example: Pendulum

A pendulum with length $L$ and position coordinates $(x, y)$ moves according to the equations

$$\begin{aligned} \dot{x} &= u & \dot{y} &= v & & \\ \dot{u} &= \lambda x & \dot{v} &= \lambda y & L^2 &= x^2 + y^2 \end{aligned}$$

Differentiating the fifth equation gives

$$0 = \dot{x}x + \dot{y}y = ux + vy$$

Differentiating a second time gives

$$\begin{aligned} 0 &= \dot{u}x + u\dot{x} + \dot{v}y + v\dot{y} \\ &= \lambda(x^2 + y^2) - gy + u^2 + v^2 \\ &= \lambda L^2 - gy + u^2 + v^2 \end{aligned}$$

and a third time

$$0 = L^2\dot{\lambda} - 3gv$$

Finally, we have derivative expressions for all variables!

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