Solutions to System Identification exam March 11, 2005.

1 a. From the definition we get:

$$\gamma^2(\omega) = \frac{|S_{yu}(i\omega)|^2}{S_{uu}(i\omega)S_{yy}(i\omega)}$$

Using the relations $S_{yu}(i\omega) = G(i\omega)S_{uu}(i\omega)$ and $S_{yy}(i\omega) = |G(i\omega)|^2 S_{uu}(i\omega) + S_{nn}(i\omega)$ and the fact that u_k is uncorrelated with n_k we obtain:

$$\begin{aligned}\gamma^2(\omega) &= \frac{|G(i\omega)|^2 S_{uu}^2(i\omega)}{S_{uu}(i\omega)(|G(i\omega)|^2 S_{uu}(i\omega) + S_{nn}(i\omega))} \\ &= \frac{1}{1 + \frac{S_{nn}(i\omega)}{|G(i\omega)|^2 S_{uu}(i\omega)}}.\end{aligned}$$

Consequently, $\gamma(\omega)$ is close to 1 when the noise is small compared to the input signal and close to 0 when the noise is large compared to the input signal.

b. By looking at the coherence function plot, we see that any model derived from this identification data will only be valid up to roughly 6Hz, or 12π radians per second. The reasons for this can be found in the input auto-spectrum S_{uu}, which shows that the input signal power drops off sharply at around 6Hz. It is likely that a controller which has been designed using this model will be poor if the closed loop bandwidth is chosen higher than 6Hz.

c. Increasing the frequency content of the input signal for frequencies above 6Hz should result in a model which is valid for higher frequencies.

2.

$$\lambda = e^{\ln 0.15/100} = 0.9812$$

3. The likelihood function is given by:

$$L(\overline{\theta}) = \prod_{k=2}^{N} f(\varepsilon_k) = \prod_{k=2}^{N} \frac{1}{\sigma\sqrt{2\pi}} e^{-\varepsilon_k^2/2\sigma^2} = \frac{1}{(\sigma\sqrt{2\pi})^{N-1}} \prod_{k=2}^{N} e^{-\varepsilon_k^2/2\sigma^2}$$

The log likelihood function is then:

$$\log L(\overline{\theta}) = -(N-1)\log(\sigma\sqrt{2\pi}) - \sum_{k=2}^{N} \frac{\varepsilon_k^2}{2\sigma^2}$$

where $\varepsilon_k = y_k - \phi_k^T \overline{\theta}$. Since the first term on the right hand side is independent of $\overline{\theta}$, we seek to maximize the second term:

$$-\sum_{k=2}^{N} \frac{(y_k - \phi_k^T \overline{\theta})^2}{2\sigma^2}$$

This is equivalent to minimizing the cost function:

$$J(\overline{\theta}) = \sum_{k=2}^{N} (y_k - \phi_k^T \overline{\theta})^2$$

which is the same as the Least Squares cost function. We can conclude that the ML and LS estimates are the same when the noise is normally distributed.

4 a. The closed loop system step response should be at least sampled 5-10 times during it's rise time. In our case we have around 100 samples during the rise time.

b. Start by checking the data manually via plots.

- remove outliers
- linear trends and non-zero mean

This should be done before the identification procedure starts.

c. The data can be divided into three parts

- data with transients
- identification data
- validation data

The amplitude should be chosen as large as possible in order to achieve a good signal-to-noise ratio and to overcome problems with friction. However, the amplitude may not be chosen larger than the range in which the linearity assumption holds. (See the section on preliminary experiments above.) Typically saturations give an upper bound on the amplitude of the input signal. The mean value is in many cases non-zero in order to reduce friction problems or to give a linearized model around a stationary point with $u^0 \neq 0$.

Let $p = 2n$ denote the number of parameters in the models $\mathcal{M}$ and N the number of data points.
The Akaike information criterion gives

$$\text{AIC}(p) = \log V(\widehat{\theta}) + \frac{2p}{N} = [0.4900 \quad 0.0400 \quad 0.0295 \quad 0.0287 \quad 0.0381]$$

The final prediction error criterion gives

$$\text{FPE}(p) = \frac{N+p}{N-p} V(\widehat{\theta}) = [1.6323 \quad 1.0408 \quad 1.0300 \quad 1.0292 \quad 1.0389]$$

The suitable number of parameters are 8.

6 a. A balanced realisation has 'balanced' observability and reachability properties, that is to say the Gramians P and Q are equal.

b. For the given state-space realization $\{\Phi, \Gamma, C\}$, direct calculations give

$$C(zI - \Phi)^{-1}\Gamma = H(z)$$

c. For a balanced realization, the asymptotic reachability Gramian P is equal to the asymptotic observability Gramian Q. The diagonal matrix $\Sigma = P = Q$ fulfills the discrete-time Lyapunov equations

$$\begin{aligned} P &= \Phi P \Phi^T + \Gamma\Gamma^T \\ Q &= \Phi^T Q \Phi + C^T C \end{aligned}$$

It can be seen that the given state-space realisation is balanced since Φ is symmetric and $\Gamma = C^T$. Solving the first equation:

$$\begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix} = \begin{bmatrix} -0.05698 & -0.1914 \\ -0.1914 & -0.643 \end{bmatrix} \begin{bmatrix} p_{11} & 0 \\ 0 & p_{22} \end{bmatrix} \begin{bmatrix} -0.05698 & -0.1914 \\ -0.1914 & -0.643 \end{bmatrix} + \begin{bmatrix} -0.9998 \\ 0.01877 \end{bmatrix} \begin{bmatrix} -0.9998 \\ 0.01877 \end{bmatrix}^T$$

gives:

$$\Sigma = P = Q = \begin{bmatrix} 1.0052 & 0 \\ 0 & 0.0634 \end{bmatrix}$$

d. Since one of the eigenvalues of Σ is much larger than the other, it is advisable to reduce the model. The state corresponding to the smaller eigenvalue will be removed by setting its dynamics to zero. By solving for x_2 and substituting into the equation for x_1 we obtain:

$$\begin{aligned} x_1(k+1) &= \left(-0.05698 + \frac{0.1914^2}{1+0.643}\right) x_1(k) + \left(-0.9998 - \frac{0.1914 \cdot 0.01877}{1+0.643}\right) u(k) \\ &= -0.0347 x_1(k) - 1.002 u(k) \end{aligned}$$

$$\begin{aligned} y(k) &= \left(-0.9998 - \frac{0.01877 \cdot 0.1914}{1+0.643}\right) x_1(k) + \frac{0.001877^2}{1+0.643} u(k) \\ &= -1.002 x_1(k) + 0.0002144 u_k \end{aligned}$$

7 a. The closed-loop system is given by

$$
\begin{aligned}
G_{cl}(s) &= \frac{\widehat{G}_P}{1 + G_C \widehat{G}_P} = \frac{(s+1)(s+3)}{(s+2)(s+3)(s+4) + (s+1)} \\
&= \frac{\frac{(s+1)}{(s+2)(s+4)}}{1 + \frac{(s+1)}{(s+2)(s+3)(s+4)}} = \frac{\frac{(s+1)}{(s+2)(s+4)}}{1 + \frac{1}{(s+3)} \frac{(s+1)}{(s+2)(s+4)}} \\
&= \frac{\frac{(s+1)}{(s+2)(s+4)}}{1 + G_C \frac{(s+1)}{(s+2)(s+4)}} \Rightarrow \widehat{G}_P = \frac{(s+1)}{(s+2)(s+4)}
\end{aligned}
$$

b. The main disadvantage with indirect identification is that any error in $G_C(s)$ (including deviation from a linear regulator, due to input saturations or anti-windup measurements) will be incorporated in directly to $\widehat{G}_P(s)$

8. Explicit factorization of the Hankel matrix:

$$
\begin{bmatrix} 1 & 0.5 & 0.25 \\ 0.5 & 0.25 & 0.125 \\ 0.25 & 0.125 & 0.0625 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \\ 0.25 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & 0.25 \end{bmatrix} = \mathcal{O}_3 \mathcal{C}_3
$$

permits extraction of

$$
\begin{aligned}
x_{k+1} &= Ax_k + Bu_k, \qquad A = 0.5, B = 1 \qquad (1) \\
y_k &= Cx_k, \qquad C = 1 \qquad (2)
\end{aligned}
$$