LUNDS UNIVERSITET

Institutionen för
REGLERTEKNIK

Automatic Control

Exam March 14 2018, 8–13

Points and Grades

Solutions and answers to all questions should be clearly motivated. The exam consists of in total 25 points.

Betyg 3: at least 12p
4: at least 17p
5: at least 22p

Allowed help

Mathematical tables (TEFYMA or similar), department's collection of formulae, calculators without preprogramming.

Results

Information about the results will be made available on the course home page.

1. Consider a control system consisting of a car driving on a road, the controller is the driver trying to maintain constant speed. Give examples of control signal, output signal, reference signal, disturbance signal. (2 p)

Solution

Control signal = gas pedal (also gear), output = speed of car, reference signal = wanted speed, disturbance = slope of road, wind, mooses,. . .

2. The relationship between input u and output y for a system is given by the differential equation

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{u}(t) + u(t)$$

a. Determine the transfer function $G(s)$. (1 p)

b. Calculate poles and zeros of $G(s)$. (1 p)

c. Write the system on state-space form $\dot{x} = Ax + Bu, \quad y = Cx + Du$, give A, B, C, D (many variants are possible, you can choose form yourself). (1 p)

Solution

a. $G(s) = \dfrac{s+1}{s^2+3s+2}$

b. Char eq $s^2 + 3s + 2 = 0$ gives poles $s = -2$ och $s = -1$. Zero in $s = -1$.

c. Controllable canoncial form with $a_1 = 3, a_2 = 2, b_1 = 1, b_2 = 1$ and $D = 0$ gives

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = [\,1 \quad 1\,], \quad D = 0$$

3. A nonlinear model for height control of a small drone is given by (4 p)

$$m\frac{d^2h(t)}{dt^2} = k\omega^2(t) - g$$
$$J\frac{d\omega(t)}{dt} + b\omega^2(t) = u(t)$$

where m, k, J, b are constants related to the drone, $h(t)$ is height, $u(t)$ motor signal and $\omega(t) \geq 0$ rotor speed and g gravitational constant.

a. Write the systemet on nonlinear state space form $\dot{x} = f(x, u)$. Use $x = [h, \dot{h}, \omega]^T$ as state vector.

b. Find all system equilibria (x^0, u^0).

c. Determine the linearised system $\frac{d}{dt}\Delta x(t) = A\Delta x(t) + B\Delta u(t)$. Does A and B depend on which equilibrium we linearise the system around ?

Solution

a. With $x_1 = h, x_2 = \dot{h}, x_3 = \omega$ we get

$$\frac{dx}{dt} = \begin{bmatrix} x_2(t) \\ \frac{k}{m}x_3^2(t) - \frac{g}{m} \\ -\frac{b}{J}x_3^2(t) + \frac{1}{J}u(t) \end{bmatrix} =: f(x,u)$$

b. $f(x,u) = 0$ gives $x_2 = 0$, $kx_3^2 = g$ och $bx_3^2 = u$, and x_1 (height) is arbitrary. Equilibria are therefore given by $(x_1^0, x_2^0, x_3^0) = (h, 0, \sqrt{\frac{g}{k}})$, and $u^0 = bg/k$.

c.

$$A = \frac{\partial f}{\partial x}\bigg|_{(x^0,u^0)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{2k}{m}\sqrt{\frac{g}{k}} \\ 0 & 0 & -\frac{2b}{J}\sqrt{\frac{g}{k}} \end{bmatrix}, \qquad B = \frac{\partial f}{\partial u}\bigg|_{(x^0,u^0)} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{J} \end{bmatrix}$$

The linearisation does not depend on the linearisation point.

4.

a. Sketch the Nyquist curve to (2 p)

$$G(s) = \frac{3}{(s+1)^3}$$

b. The system $Y(s) = G(s)U(s)$ is controlled by proportional feedback $u = K(r - y)$. For what $K \geq 0$ is the closed loop system asymptotically stable?

(You don't have to use the Nyquist diagram in a, if you prefer using another method). (2 p)

Solution

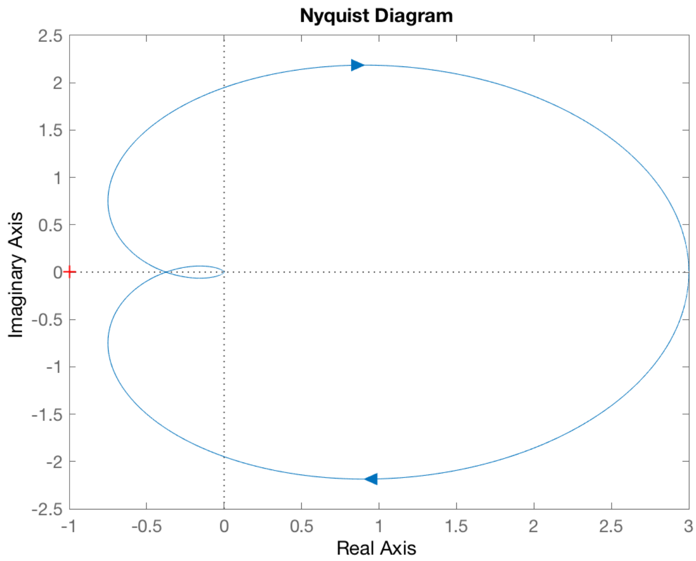
a. Amptlidue is given by $|G(i\omega)| = \frac{3}{(\omega^2+1)^{3/2}}$ and phase by $\arg G(i\omega) = -3\,\mathrm{atan}(\omega)$. Amplitude therefore decreases from 3 to 0 when frequency goes from 0 to ∞ and the phase from 0 to -270 grader. Of special interest is the point where the curve intersects the negative real axis. This occurs when $-3\,\mathrm{atan}(\omega) = -\pi$, dvs då $\omega = \tan(\pi/3) = \sqrt{3}$, giving $|G| = 3/(3+1)^{3/2} = 3/8$, i.e. the intersection is in at $-3/8$. The resulting Nyquist diagram is given in Figure1.

b. We can use the Nyquist curve and concludethat the system is as. stable when $0 \leq K < 8/3$.

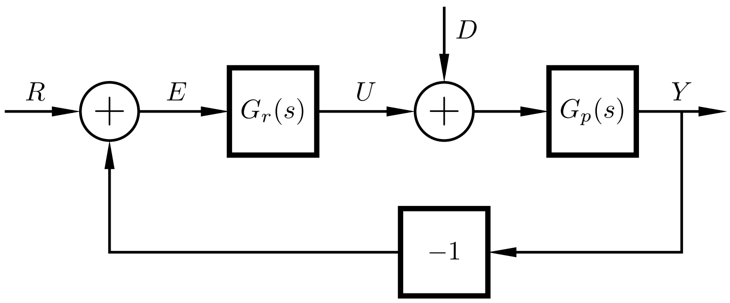
Alternatively we can calculate the closed loop system

$$\frac{G_0 K}{1 + G_0 K} = \frac{\frac{3K}{(s+1)^3}}{1 + \frac{3K}{(s+1)^3}} = \frac{3K}{s^3 + 3s^2 + 3s + 1 + 3K}$$

and use the stability condition for 3rd order systems giving $1 + 3K > 0$ och $3 \cdot 3 > 1 + 3K$, hence $-1/3 < K < 8/3$.



Figur 1 Nyquist curve in 4.



Figur 2 Control System in Problem 5

5. The process $G_p(s) = \frac{1}{(s+1)^3}$ is controlled with $G_r(s) = K > 0$, see Figure 5. (4 p)

a. Determine stationary error $\lim_{t\to\infty} e(t)$ when $R = 0$ and $D(s) = 1/s$ (step disturbance). Give the result for a general constant $K > 0$.

b. Can a value K be found that gives stationary error smaller than n 0.01, i.e. $|\lim_{t\to\infty} e(t)| < 0.01$?

c. Suggest a better controller giving an as. stable closed loop system with zero stationarly error $\lim_{t\to\infty} e(t) = 0$. (Only name on controller structure is not enough a brief motivation for why it works is needed.)

Solution

a. We get $E(s) = \frac{-G(s)}{1+G(s)K} D(s)$ giving

$$sE(s) = \frac{-sG(s)}{1+G(s)K} 1/s = \frac{-1}{s^3+3s^2+3s+1+K}$$

The system is stable when $1 + K > 0$ and $3 \cdot 3 < 1 + K$, giving $-1 < K < 8$. For such K we can use the final value theorem, which gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{-1}{s^3 + 3s^2 + 3s + 1 + K} = \frac{-1}{1+K}$$

b. The condition is fulfilled when $K > 99$ but these values give unstable systems, hence such a K can not be found.

c. An I-controller $G_r(s) = K/s$ gives loop gain $G_0 = \frac{K}{s(s+1)^3}$. A sketch of Bode diagram or Nyquist curve shows that the closed loop system is stable for small $K > 0$. The integrator in the controller ensures that the stationary error becomes zero. Also a PI or PID controller could be used, by similar reasoning.

6. A process is given by

$$G_P(s) = \frac{s+2}{(2s+1)^3}.$$

It is used with the controller

$$G_R(s) = 4$$

which gives an as. stable closed loop system.

a. Determine the sensitivy function $S(s)$. (1 p)

b. How much is low frequency disturbances attenuated by closing the loop (i.e. comparing closed loop with open loop)? (1 p)

Solution

a.

$$S(s) = \frac{1}{1 + G_P G_R} = \frac{(2s+1)^3}{(2s+1)^3 + 4(s+2)}$$

b. Since $S(0) = 1/9$ the disturbance is attenuated a factor 9.

7. Determine if the following statements are true or false. (2 p)

$$G(s) = \frac{s^m + b_1 s^{m-1} + \ldots + b_m}{s^n + a_1 s^{n-1} + \ldots + a_n}$$

A) If $a_k > 0$ for all k then the system is stable
B) If $a_k > 0$ for all k then the system is asymptotically stable
C) If $a_k < 0$ for some k then the system is unstable
D) If $a_k = 0$ for some k then the system is unstable

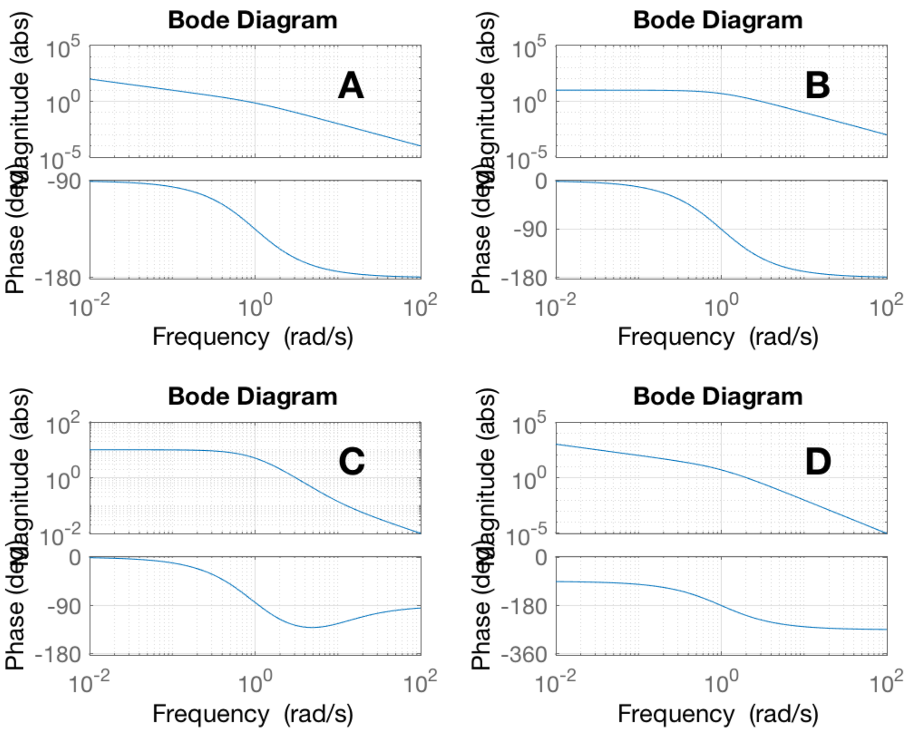
Motivation needed. Give counter examples to false statements.

Solution

A,B,D false (third orders systems are counter examples to A,B, en integrator to D). C is true.

8. The Bode diagrams A-D in Figure 3 show the loop gains $G_0 = G_p G_{R,i}$ when the process $G_p(s) = 1/(s+1)^2$ is controlled with the following controllers

$$G_{R,1}(s) = 10 \qquad \text{P-regulator}$$
$$G_{R,2}(s) = \frac{10}{s} \qquad \text{I-regulator}$$
$$G_{R,3}(s) = 1 + \frac{1}{s} \qquad \text{PI-regulator}$$
$$G_{R,4}(s) = 10 + s \qquad \text{PD-regulator}$$



Figur 3 Bode diagrams A-D for $G_0 = G_P G_{R,i}$ in Problem 8.

a. Pair the loop gains i A-D with the controllers 1-4. (2 p)

b. Which Bode diagram A-D gives an unstable closed loop system? (1 p)

c. What is the phase margin approximately in Bode diagram A? (1 p)

Don't forget to motivate your answers.

Solution

a. The process G_p has a Bode diagram with slope 0 and phase 0 for low frequency, and slope -2 and phase -180 degrees for high frequenciess. The Bode-diagrammet for $G_0 = G_p G_R$ is for the different controllers :

Case 2 and 3, with I-part, can be recognizerd since the phase of G_0 starts at -90 degrees, hence A and D. For high frequencies case 2 will get slope -3 and phase -270 degrees, therefore 2=D, and hence 3=A.

Case 1 with P-controller gives $G_0 = 10/(s+1)^2$ which is recognized in figure B.

Case 4 with PD-control is the only with slope -1 and phase -90 degrees for high frequencies, therefore 4=C.

Svar: 1-B, 2-D, 3-A, 4-C

b. Figure D is the only case where the phase is below -180 degrees. We see that the amplitude is larger than -1 when this happens. Hence case D gives an unstable closed loop.

c. We see that the amplitude becomes 1 slightly to the left of $\omega = 1$, at about $\omega = 0.8$. The phase is then seen to be roughly -130 degrees, so phase margin is about 50 degrees. A closer calculation, using that we know the controller is given by case 3 gives:

$$|G_0(i\omega)| = |G_p(i\omega)||G_{R,3}(i\omega)| = \frac{1}{1+\omega^2}\sqrt{1+\frac{1}{\omega^2}}$$

Cutoff frequency is determined by $|G_0(i\omega_c)| = 1$ given by $1 = \frac{1}{\omega_c\sqrt{1+\omega_c^2}}$ which gives $\omega_c = 0.786$. We get phase margin

$$\varphi_m = \pi + \arg(G_p(i\omega_c)G_{R,3}(i\omega_c))$$

which with $\arg(G_p(i\omega_c)) = -2\arctan(\omega_c)$ and $\arg(G_{R,3}(i\omega_c)) = -\arctan\frac{1}{\omega_c}$ gives

$$\varphi_m = 3.142 - 1.332 - 0.905 = 0.905 \text{ rad} = 51.8 \text{ degrees}.$$