LUNDS UNIVERSITET

Institutionen för
REGLERTEKNIK

Automatic Control

Exam March 16 2016 kl 8–13

Points and Grades

Solutions and answers to all questions should be clearly motivated. The exam consists of in total 25 points.

Betyg 3: at least 12p
4: at least 17p
5: at least 22p

Allowed help

Mathematical tables (TEFYMA or similar), collection of formulae, calculators without preprogramming.

Results

Will be presented in LADOK before March 31. More info on course home page.

1. Consider the system $Y(s) = G(s)U(s)$ where

$$G(s) = \frac{s}{s^2 + 2s + 1}$$

a. What are the poles and zeros? Is the system stable? (1.5 p)

b. Find a differential equation that relates $u(t)$ and $y(t)$. (0.5 p)

c. What is the output $y(t)$ if $u(t)$ is a unit step function (the system is at rest when $t = 0$)? (1 p)

Solution

a. Poles: -1 (double).Zero: 0. The system is stable.

b.

$$y'' + 2y' + y = u'$$

c. The output signal is $\dfrac{s}{(s+1)^2}\dfrac{1}{s} = \dfrac{1}{(s+1)^2}$. Using inverse Laplace transform and table lookup, we have $y(t) = te^{-t}$).

2. The Bode plot of an open loop stable system is given in Figure 1. Study the plot and answer whether the following statements are false or true. Each correct answer with motivation gives 0.5 p. (3 p)

a. The static gain of the process is 10^{-3}.

b. The process contains an integrator.

c. The process has one pole and one zero.

d. The process has a time delay.

e. With a P-controller $u = -y$ the closed loop will be stable.

f. With a P-controller $u = -100y$ the closed loop will be stable.

Solution

a. False. As seen from the Bode plot the static gain of the system is 0.01.

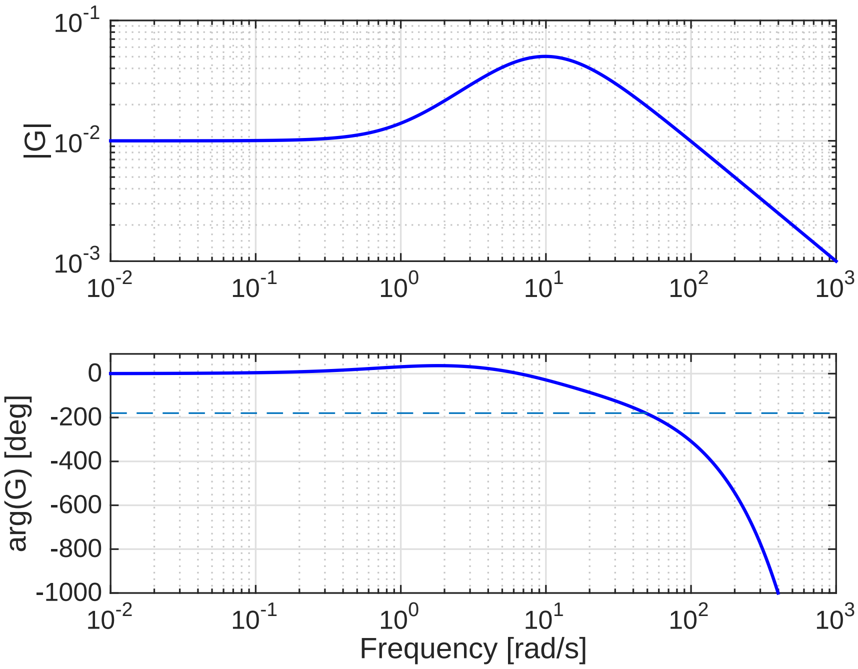
b. False. As the low frequency gain does not go towards infinity, we can conclude that the system has no integrator.

c. False. The system has one zero at 1 and two poles at 10.

d. True. We can see that the phase of the system goes towards minus infinity at high frequencies, thus we can conclude that the system has a time delay.

e. True. The loop gain is then always smaller than one, and there can be no encirclement of -1. According to the (simplified) Nyquist criterion the closed loop will be stable.

f. False. The cutoff frequency will then be around $\omega_c = 100$ where the phase is below -180 degrees, which gives a negative phase margin.



Figur 1 Bode plot for Problem 2

3. Are the following statements **a-d** true or false? Motivate. (2 p)

a. At the frequencies where the sensitivity function has gain larger than 1 the Nyquist curve for the open system will have a distance smaller than 1 to the point -1.

b. A lag compensator link gives a non-positive phase contribution at all frequencies.

c. When the process $\frac{s}{s^2+2s+1}$ is controlled by a PI controller there will be no stationary errors after a step change in the reference value.

d. When a P-controller is designed according the the Ziegler-Nichols frequency response (self-oscillation) method the resulting closed loop system will have the gain margin $A_m = 2$.

Solution

a. True. The distance from the open loop systems Nyquist curve to the point -1 is

$$|1 + G_0(i\omega)| = \frac{1}{|S(i\omega)|}$$

where G_0 is the loop gain and S the sensitivity function. Hence $|S(i\omega)| > 1 \implies |1 + G_0(i\omega)| < 1$.

b. True. The phase for a lag compensator is

$$arg(M\frac{1+\frac{i\omega}{a}}{1+\frac{i\omega M}{a}}) = arctan(\frac{\omega}{a}) - arctan(\frac{\omega M}{a})$$

Since $\omega \geq 0,\ a > 0,\ M > 1$ the lag compensator has a non-positive phase.

c. False. The zero in the orgin cancels the integrator of the PI controller. Therefore the loop gain does not have an integrator, which is needed for the stationary error to become zero after a step change.

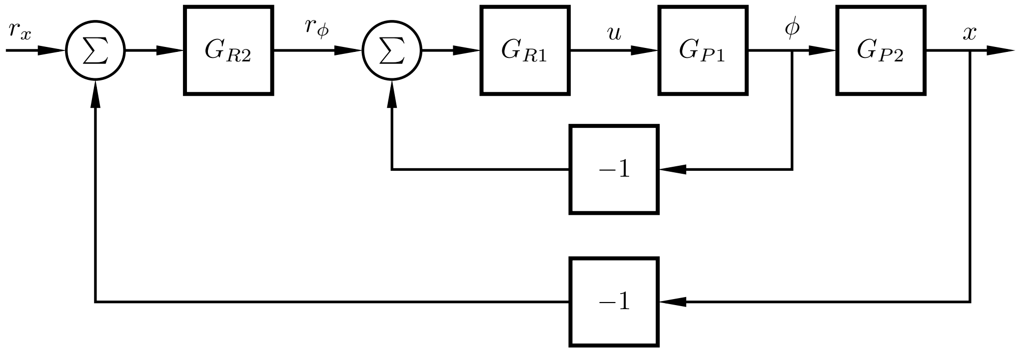
d. True. The choice $K = 0.5K_0$ gives a gain margin of $A_m = 2$.

4. The goal for the process "Ball and Beam" is to control the position of a ball x along a beam by controlling the beam angle by a motor. The control signal u controls the change rate of the angle. One can measure both the ball position x and beam angle ϕ. To solve the control problem one has chosen the structure in Figure 2.

A linear approximation of the process gives the transfer functions

$$G_{P1} = \frac{5}{s}, \qquad G_{P2} = \frac{10}{s^2}$$

The transfer functions G_{R1} and G_{R2} should be determined.



Figur 2 The control structure in Problem 4.

a. What is the name of the control structure in Figure 2? (0.5 p)

b. Calculate a P-controller $G_{R1} = K$ that places the pole of the closed loop system from r_ϕ to ϕ in -10. (1 p)

c. The controlled system from r_ϕ to ϕ can be approximated with its static gain as long as the inner loop control is significantly faster than the control of the outer loop. Use this approximation and calculate a PD-controller $G_{R2} = K(1 + sT_d)$ that places the closed loop poles in -1. (1.5 p)

d. After having implemented the design on the real process one tries to make the inner loop control of the beam angel even faster by increasing the value of K in the P-controller. With large values of K the beam starts vibrating randomly and oscillating, something the model can not explain. What can be the reason for this? (1 p)

Solution

a. Cascade control.

b. The transfer function from r_ϕ to ϕ is

$$G_{r_\phi \to \phi}(s) = \frac{5K}{s + 5K}$$

The value $K = 2$ moves the pole to -10.

c. The approximation gives

$$G_{r_\phi \to \phi}(s) \approx G_{r_\phi \to \phi}(0) = 1$$

With this approximatoin the design of G_{R2} can be done as if the inner loop did not exist. The closed loop system is then given by

$$G_{r_x \to x}(s) \approx \frac{10K(sT_d + 1)}{s^2 + 10KT_d s + 10K}$$

Identification with the wanted characteristic polynomial $(s+1)^2 = s^2 + 2s + 1$ gives

$$\begin{aligned} K &= 0.1 \\ T_d &= 2 \end{aligned}$$

d. The real process has measurmement noise. With larger values of K this results in large random variations in the control signal due to the random noise, which was not modeled. Other explanations include unmodelled dynamics such as time delays and resonant mechanical modes in the flexible beam.

5. A system is described by the nonlinear differential equation

$$\ddot{z} + \frac{2\dot{z}}{(1+z^2)^2} - z = \sqrt{u}.$$

where $u \geq 0$ is the input and the output is given by $y = z^2 + u^2$.

a. Introduce the states $x_1 = z$ and $x_2 = \dot{z}$ and determine a state space realisation of the system. (1 p)

b. Determine all stationary points (x_0, u_0). (1 p)

c. Linearise the system around the equilibrium point corresponding to $u_0 = 4$. (2 p)

Solution

a. We get

$$\begin{aligned} \dot{x}_1 &= x_2 && (= f_1(x,u)) \\ \dot{x}_2 &= -\frac{2x_2}{(1+x_1^2)^2} + x_1 + \sqrt{u} && (= f_2(x,u)) \\ y &= x_1^2 + u^2 && (= g(x,u)) \end{aligned} \tag{1}$$

b. From the first equation we have $x_2^0 = 0$. Putting $x_2 = 0$ in the second state equation gives the following condition for statoinarity

$$0 = x_1 + \sqrt{u}. \tag{2}$$

Stationary points are given by $(x_1^0, x_2^0, u^0) = (-\sqrt{t}, 0, t)$, $t \geq 0$. In stationarity the output is hence given by $y^0 = t + t^2$.

c. $u = 4$ gives the stationary point $(x_1^0, x_2^0, u^0, y^0) = (-2, 0, 2, 20)$. The partial derivatives are

$$\frac{\partial f_1}{\partial x_1} = 0, \qquad \frac{\partial f_1}{\partial x_2} = 1, \qquad \frac{\partial f_1}{\partial u} = 0,$$
$$\frac{\partial f_2}{\partial x_1} = 8\frac{x_2 x_1}{(1+x_1^2)^3} + 1, \qquad \frac{\partial f_2}{\partial x_2} = \frac{-2}{(1+x_1^2)^2}, \qquad \frac{\partial f_2}{\partial u} = \frac{1}{2\sqrt{u}}$$
$$\frac{\partial g}{\partial x_1} = 2x_1, \qquad \frac{\partial g}{\partial x_2} = 0, \qquad \frac{\partial g}{\partial u} = 2u,$$

Introduce new variables

$$\begin{aligned} \Delta x &= x - x^0 \\ \Delta u &= u - u^0 \\ \Delta y &= y - y^0. \end{aligned} \tag{3}$$

The linearised system is given by

$$\begin{aligned} \frac{d}{dt}\Delta x &= \begin{bmatrix} 0 & 1 \\ 1 & -\frac{2}{25} \end{bmatrix} \Delta x + \begin{bmatrix} 0 \\ \frac{1}{4} \end{bmatrix} \Delta u \\ \Delta y &= \begin{bmatrix} -4 & 0 \end{bmatrix} \Delta x + 8\Delta u \end{aligned} \tag{4}$$

6. A system is given by the state space form (4 p)

$$\begin{cases} \dot{x} = \begin{bmatrix} -4 & 2 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

a. Determine the poles and zeros of the system.

b. Is the system controllable ?

c. Introduce state feedback $u = -Lx + l_r r$ with $L = \begin{bmatrix} 0 & 4 \end{bmatrix}$. Determine the poles of the closed-loop system. Also determine l_r so that closed loop system has unit stationary gain from r to y.

Solution

a. The transfer function of the system is given by
$G(s) = C(sI - A)^{-1}B$
$= \frac{1}{(s+4)(s+3)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+3 & 2 \\ 0 & s+4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{s+5}{(s+4)(s+3)}$
Thus the system has one zero at -5 and two poles at -4 and -3.

b. The given L matrix is $\begin{bmatrix} 0 & 4 \end{bmatrix}$ and $l_r = 1$. Substituting these into u, we get $u = 4x_2 + r$. Therefore

$$\begin{aligned} G_{cl}(s) = C(sI-(A-BL))^{-1}Bl_r &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+4 & 2 \\ 0 & s+7 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} l_r \\ &= \frac{1}{(s+4)(s+7)} \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s+7 & -2 \\ 0 & s+4 \end{bmatrix} l_r \begin{bmatrix} 1 \\ 1 \end{bmatrix} l_r \\ &= \frac{s+5}{(s+4)(s+7)} l_r. \end{aligned}$$

Thus the system has two poles at -4 and -7 and a zero at -5. Evaluating the stationary gain gives the equation $\frac{5}{28}l_r = 1$ hence $l_r = \frac{28}{5}$.

7. You want to control a process with the transfer function

$$G(s) = \frac{10}{s^2}$$

so that the system gets crossover frequency $\omega_c = 10$ rad/s and phase margin 30 degrees. Design an appropriate compensation link that achieves this. (3 p)

Solution

We choose a lead compensator

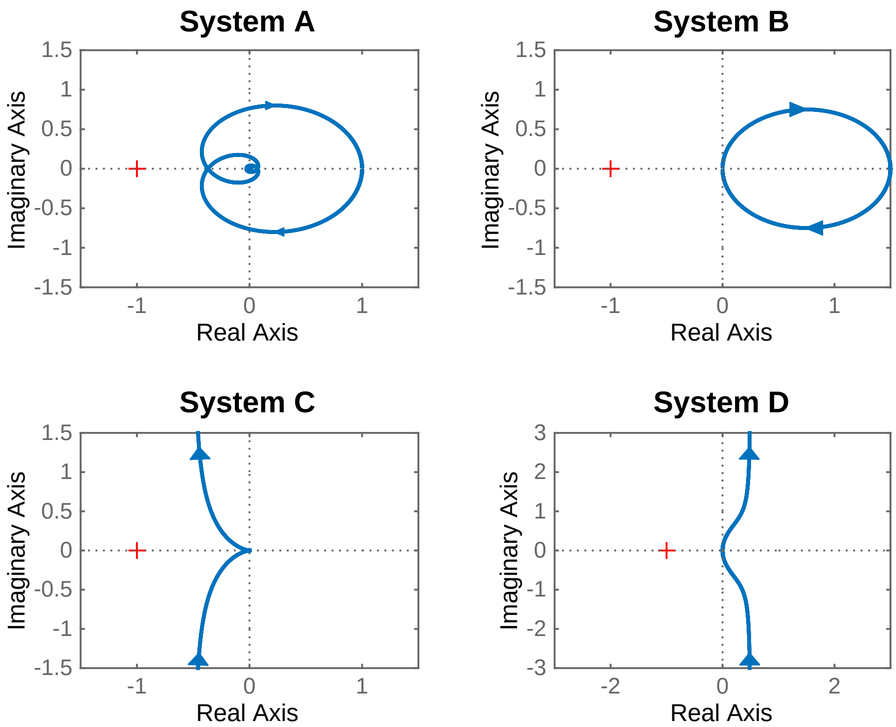
$$G_k(s) = K_k \frac{1 + s/b}{1 + s/(bN)}.$$

Since G has phase -180 degrees for all frequencies we need 30 degrees phase advance. From the diagram in the collection of formulae we see that we should have $N = 3$. We should ahve $\omega_c = b\sqrt{N}$, hence $b = 10/\sqrt{2} = 5.77$. Finally the condition $G(i\omega_c)G_k(i\omega_c) = 1$ gives

$$10/10^2 K_k \sqrt{N} = 1$$

hence $K_k = 5.77$. The controller is

$$G_k(s) = 5.77\frac{1 + s/5.77}{1 + s/17.3} = \frac{100s + 577}{5.77s + 100}.$$



Figur 3 Nyquist diagrams A-D in Problem 8. The diagram is drawn for both positive and negative frequencies.

8. The Nyquist diagrams A-D of four systems are given in Figure 3. The systems are given by the four transfer functions below. Find which diagram corresponds to which transfer function and motivate your answer. (2 p)

$$G_1(s) = \frac{3}{s+2}, \quad G_2(s) = \frac{2}{s(s+2)}, \quad G_3(s) = \frac{e^{-s}}{(s+1)^2}, \quad G_4(s) = \frac{1+s}{s(1+s/2)}$$

Solution

We immediately recognize the first order system G_1 in diagram B and the system G_3 with time delay is in A. To distinguish between the two other systems, both containing an integrator, one can check that G_2 has phase below -90 degrees and G_4 above -90 degrees (G_4 is an integrator with phase advance compensation). Therefore G_2 is C and G_4 is D. Answer: 1234=BCAD.