LUNDS TEKNISKA HÖGSKOLA
Lunds universitet

Institutionen för
REGLERTEKNIK

Automatic Control

Exam December 16, 2013, 8.00-13.00

Points and grading

Solutions and answers to all problems should be clearly motivated. The exam consists of 25 points. The points on different problems are clearly marked.
Grade limits:

3: 12 points

4: 17 points

5: 22 points

Allowed Aids

Mathematical tables, the "collection of formulae" and (not pre-programmed) calculators. The book is not allowed.

Exam Results

Will be put in Ladok by January 3 2014. See also the course home page for more information.

Good luck and Happy Holidays!

Solutions to exam in Automatic Control 2013-Dec-16

1. Consider the system

$$\dot{x} = \begin{pmatrix} -5 & 0 \\ 1 & 3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & 1 \end{pmatrix} x.$$

a. Determine the transfer function $G(s)$. (2 p)

b. Calculate the poles of the system. Is the system unstable, (marginally) stable or asymptotically stable? (1 p)

Solution

a. The transfer function from u to y is given by $G(s) = C(sI - A)^{-1}B$, ie

$$G(s) = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} s+5 & 0 \\ -1 & s-3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{(s+5)(s-3)}.$$

b. The poles to are given by the roots of the denominator polynomial $(s+5)(s-3)$, ie by $s = -5$ and $s = 3$. Since one pole is in the right half plane the system is unstable.

2. The relation between input u and output y for a system is given by (2 p)

$$y(t) = u(t-3).$$

Calculate the gain and phase for the transfer function from u to y and sketch the Bode diagram for the interval $\omega \in [0.1, 10]$. Give the phase for $\omega = 0.1$ and $\omega = 10$ expressed in degrees.

Solution The transfer function for a time delay is $G(s) = e^{-sL}$ in thsi case $L = 3$. The Bode diagram is given in the collection of formulae. The amplitude is $|G(i\omega)| = 1$ for all frequencies and the phase is given by $\arg G(i\omega) = -3\omega$ radians $= -180 \cdot 3\omega/\pi$ grader. For $\omega = 0.1$ the phase is fasen $-54/\pi \approx -17$ degrees, ad for $\omega = 1$ it is ≈ -1720 degrees.

3. A basic PID-controller is given by the control law

$$u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)\mathrm{d}\tau + T_d \frac{\mathrm{d}e(t)}{\mathrm{d}t}\right),$$

where $e(t) = r(t) - y(t)$ is the difference between reference value $r(t)$ and measurement signal $y(t)$.

a. Show that the controller transfer function is (1 p)

$$G_R(s) = K\left(1 + \frac{1}{sT_i} + sT_d\right).$$

b. In many practical implementations of a PID-controller the following slightly modified control law is used

$$u(t) = K\left(e_p(t) + \frac{1}{T_i}\int_0^t e(\tau)\mathrm{d}\tau + T_d\frac{\mathrm{d}e_d(t)}{\mathrm{d}t}\right),$$

where $e_p(t) = br(t) - y(t)$ with constant $b \in [0, 1]$, and $e_d(t) = -y(t)$. What advantages are there with this variant of PID-controller? (1 p)

Solution

a. Follows since the Laplace transform of $\int_0^t e$ is $\frac{E(s)}{s}$ and of $\frac{de}{dt}$ is $sE(s)$.

b. Both modificatins can lead to better behavior for changes in the reference signal. By adjusting the factor b one can influence the speed for responses to changes in reference value, which then need not be the same as the speed when compensating for an output disturbance. One can for instance sometimes avoid overshoot in the response to step changes in the reference. By not derivating the reference signal one can also avoid large control jumps when the reference is changed rapidly.

4. Describe, preferably by a practical example, the phenomenon of windup during PID control. Also describe a remedy. (2 p)

Solution See the book.

5. Consider the following nonlinear differential equations

$$\dot{x}_1 = -x_1 + u^2$$
$$\dot{x}_2 = -x_1 + \sqrt{x_2}.$$

a. Assume $u^0 = 1$ and find the stationary points of the system. (0.5 p)

b. Linearize the system around the stationary point you found in **a.** (1.5 p)

Solution

a. Stationary points are determined by setting $\dot{x}_1 = \dot{x}_2 = 0$, which gives

$$0 = -x_1^0 + 1^2$$
$$0 = -x_1^0 + \sqrt{x_2^0},$$

from which we get the stationary point $(u^0, x_1^0, x_2^0) = (1, 1, 1)$.

b. Calculation of partial derivatives of the two equations (denoted f_1 and f_2) gives

$$\frac{\partial f_1}{\partial x_1} = -1, \qquad \frac{\partial f_1}{\partial x_2} = 0, \qquad \frac{\partial f_1}{\partial u} = 2u,$$
$$\frac{\partial f_2}{\partial x_1} = -1, \qquad \frac{\partial f_2}{\partial x_2} = \frac{1}{2\sqrt{x_2}}, \qquad \frac{\partial f_2}{\partial u} = 0$$

Putting in the stationary point $(u^0, x_1^0, x_2^0) = (1, 1, 1)$ gives

$$\frac{\partial f_1}{\partial x_1}(1,1,1) = -1, \qquad \frac{\partial f_1}{\partial x_2}(1,1,1) = 0, \qquad \frac{\partial f_1}{\partial u}(1,1,1) = 2,$$
$$\frac{\partial f_2}{\partial x_1}(1,1,1) = -1, \qquad \frac{\partial f_2}{\partial x_2}(1,1,1) = 0.5, \qquad \frac{\partial f_2}{\partial u}(1,1,1) = 0$$

and if we introduce new variables

$$\Delta x = x - x^0$$
$$\Delta u = u - u^0$$

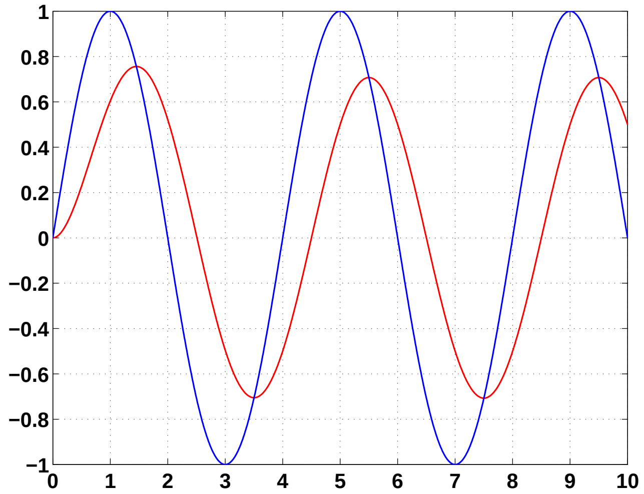
we get the linearized system

$$\Delta \dot{x} = \begin{bmatrix} -1 & 0 \\ -1 & 0.5 \end{bmatrix} \Delta x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} \Delta u.$$

6. Figure 1 shows input $u(t) = \sin(\omega t)$ and output $y(t)$ for the system

$$G(s) = \frac{a}{s + a}.$$

Determine the parameters ω and a. (2 p)



Figur 1 Input and output in Problem 6

Solution The input has amplitude 1 and is therefore the curve with largest amplitude. From the figure we see that period time is $T = 4$ which gives $\omega = 2\pi/T = \pi/2$. The output after transients have decayed is given by

$$y(t) = |G(i\omega)| \sin(\omega t + \arg G(i\omega)).$$

From the figure we see that $|G(i\omega)| \approx 0.7$ and since the zero crossing are delayed by 0.5 time units, the phase is $\arg G(i\omega) = -0.5/T \cdot 2\pi = -\pi/4$. Comparing with formulas for first order systems : $|G(i\omega)| = \frac{a}{\sqrt{\omega^2+a^2}}$ or alternatively $\arg G(i\omega) = -\arctan(\omega/a)$ we see that $a = \omega$. Answer: $\omega = a = \pi/2$.

7. The height control system for Sant-Claus sleigh is given by the equations

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

with $u = -\begin{pmatrix} 2 & 2 \end{pmatrix} x + 3r$.

a. Find the poles of the closed loop systems. (2 p)

b. Santa-Claus' elphes have worked hard with trying to find a controller placing the poles in $s = -10$ but they have failed. Can you explain why? (1 p)

c. Santa can not measure both x_1 and x_2, only the signal y, he therefore would like a Kalman filter. Design such, so that the Kalman filter poles become $s = -8$. (2 p)

Solution

a. The poles are the eigenvalues of

$$A - BL = \begin{pmatrix} -1 & 1 \\ 0 & -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 2 & 2 \end{pmatrix} = \begin{pmatrix} -3 & -1 \\ 0 & -3 \end{pmatrix}$$

Since the matrix is triangular, the eigenvalues are given by the diagonal elements, ie there are two poles at $s = -3$.

b. The system is uncontrollable, which can be seen by calculating the controllability matrix

$$W = \begin{pmatrix} B & AB \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

which is not invertible. (State x_2 is uncontrollable.)

c. An estimator is given by

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x}).$$

The characteristic polynomial for the Kalman filter is given by

$$\det(sI - A + KC) = \left| \begin{pmatrix} s+1+k_1 & -1 \\ k_2 & s+3 \end{pmatrix} \right| = (s+1+k_1)(s+3)+k_2 = (s+8)^2.$$

Identification of coefficients give $k_1 = 12, k_2 = 25$.

8. A process is given by $G_p(s) = \frac{1}{s+1}$ and is controlled by a PI-controller given by $G_{PI}(s) = 1 + \frac{2}{s}$.

a. When the reference is a ramp $r(t) = at$ one gets a stationary error, determine the size of this error. (2 p)

b. Introduce then a compensation link $G_K(s)$ that decreases the stationary ramp error a factor 10 without harming the system stability noticably (6 graders degrees smaller phase margin is acceptable). The loop transfer function is hence now $G_0(s) = G_p(s)G_{PI}(s)G_K(s)$. (2 p)

Solution

a. We get

$$sE(s) = \frac{s}{1 + G_p(s)G_{PI}(s)}R(s) = \frac{s}{1 + \frac{1}{s+1}(1 + \frac{2}{s})}\frac{a}{s^2} = \frac{a(s+1)}{s(s+1)+s+2}.$$

Stationary error is given by $\lim_{s\to 0} sE(s)$ since the poles of $sE(s)$ are in the left half plane (the polynomial $s(s+1)+s+2$ is of degree 2 and has positive coefficients) so the final value theorem can be used and gives

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{a}{2}.$$

b. We introduce a lag compensator $G_K(s) = \frac{s+a}{s+a/M}$. Since the initial loop has integral actoin, the stationary error decreases a factor M, we therefore put $M = 10$. To use the rule of thumb $a = 0.1\omega_c$ we must determine the cut off frequency ω_c given by $|G_p(i\omega_c)G_{PI}(i\omega_c)| = 1$. We get

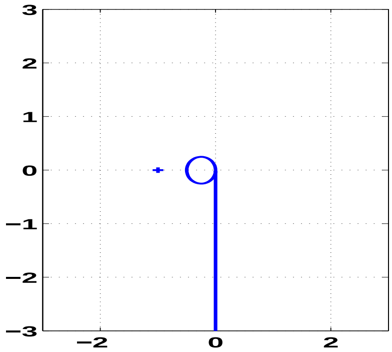
$$1 = \left| \frac{1}{i\omega_c + 1}\frac{i\omega_c + 2}{i\omega_c} \right| = \frac{(\omega_c^2 + 4)^{1/2}}{(\omega_c^2 + 1)^{1/2}\omega_c}$$

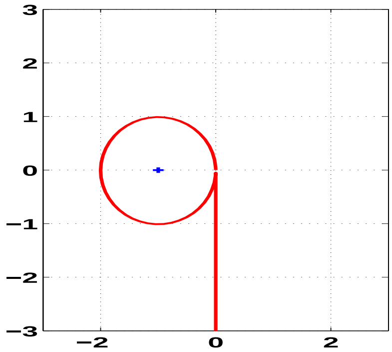
which gives the equation $\omega_c^2(\omega_c^2 + 1) = \omega_c^2 + 4$, from which $\omega_c = \sqrt{2}$. We hence choose $a = 0.1 \cdot \sqrt{2} \approx 0.14$.

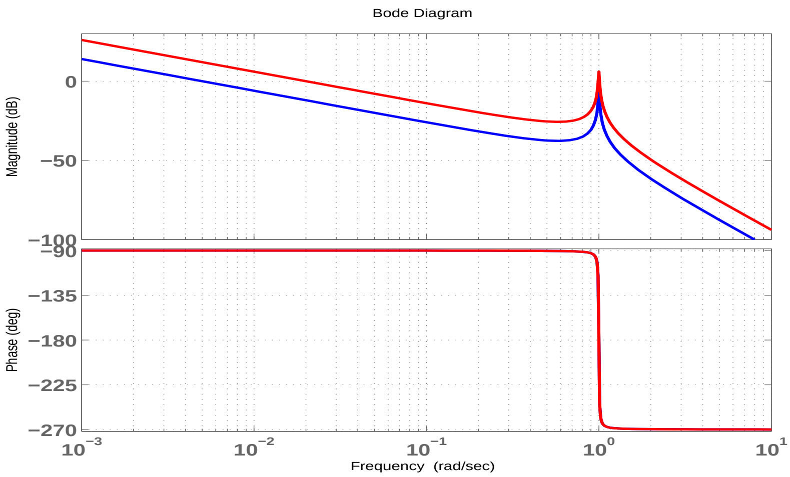
9. An atomic force microscope controlled by an I-controller is described by the loop transfer function

$$G_0(s) = G_R(s)G_p(s) = \frac{k_i}{s}\frac{\omega_0^2}{s^2 + 2\omega_0\zeta s + \omega_0^2}$$

whose Nyquist curve and Bode diagram is shown beow for $\omega_0 = 1, \zeta = 0.005$ for two different values of k_i.







a. Show that the intersection between the Nyquist curve and the negative real axis is given (when $k_i > 0$) by $G_0(i\omega_0) = -\frac{k_i}{2\zeta\omega_0}$. (1 p)

b. Find k_i giving the amplitude margin $A_m = 2$. (You can use the information in problem a.) (1 p)

c. Assume $\omega_0 = 1, \zeta = 0.005$ as in the figures. What phase margin is obtained for a k_i giving amplitude margin $A_m = 2$? (Reading in diagram is ok.) (1 p)

Solution

a. If we use $s = i\omega_0$ we see that

$$G(i\omega_0) = \frac{k_i}{i\omega_0}\cdot\frac{\omega_0^2}{-\omega_0^2 + 2i\omega_0^2\zeta + \omega_0^2} = -\frac{k_i}{2\zeta\omega_0}$$

which is hence on the negative real axis. It is easy to see that there is no other intersectin since the phase decreases monotonously from -90 to -270 degrees.

b. We have $-\frac{1}{A_m} = -\frac{k_i}{2\zeta\,\omega_0}$ which gives $k_i = \zeta\,\omega_0$ when $A_m = 2$.

c. The left Nyquist curve has amplitude margin 2. From the figure we see that the phase maring is approximately 90 degrees.