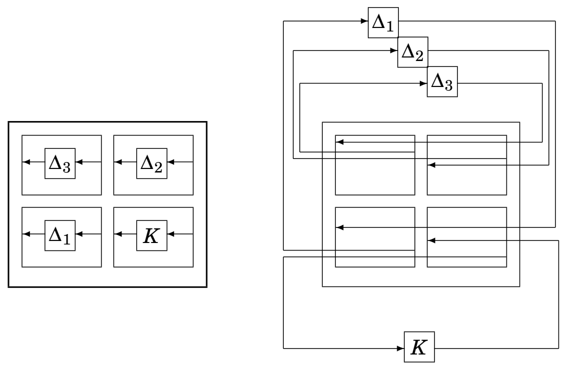
Lecture 4

- Structured Uncertainties.
- Structured Singular Value μ.
- Structured Robust Stability.
- Structured Robust Performance.
- μ Synthesis via $D-K$ iterations.
- Integral Quadratic Constraints

Pulling out Uncertainties



Structured Uncertainty

The new pulled out uncertainty has a diagonal structure composed of primitive uncertain blocks. Every primitive block can be

- complex unstructured matrix uncertainty to represent neglected dynamics.
- real parameter scalar uncertainty to represent uncertainty in system coefficients.

Usually real uncertainty is much harder to deal with. One (conservative) way to treat it is to cover it with complex uncertainty.

Thus we shall assume that

$$\Delta(s) = \text{diag}\,\{\delta_1(s)I_{r_1}, \ldots, \delta_K(s)I_{r_K}, \Delta_1(s), \ldots, \Delta_L(s)\}$$

where δ_k, $\Delta_l \in RH_\infty$ and $\|\delta_k\|_\infty \le 1$, $\|\Delta_l\|_\infty \le 1$.

Structured Singular Value

Recall the Small Gain Theorem which says that $(I - M\Delta)^{-1} \in RH_\infty$, $\forall \Delta \in BRH_\infty$ iff $\|M\|_\infty < 1$.

Thus if there exist a frequency ω and a complex matrix Δ such that

$$\det(I - M(j\omega)\Delta) = 0$$

then $\|\Delta\|$ is an upper bound on the stability margin $\|M\|_\infty^{-1}$.

Given a matrix $M \in C^{p\times q}$ introduce

$$\alpha_{\min} = \inf\{\|\Delta\| \ : \ \det(I - M\Delta) = 0, \ \Delta \in C^{q\times p}\}.$$

We have the relation

$$\|M\| = \sigma_{\max}(M) = \frac{1}{\alpha_{\min}}.$$

Now consider the structured uncertainty set

$$\mathbf{D} \ = \ \{\text{diag}\,[\delta_1 I_{r_1}, \ldots, \delta_K I_{r_K}, \Delta_1, \ldots, \Delta_L] \ : \quad \delta_k \in C, \ \Delta_l \in C^{m_l\times m}$$

Definition: Given a matrix $M \in C^{n\times n}$ the *structured singular value* $\mu_{\mathbf{D}}(M)$ is defined as

$$\mu_{\mathbf{D}}(M) =: \frac{1}{\min\{\|\Delta\| \ : \ \det(I - M\Delta) = 0, \ \Delta \in \mathbf{D}\}}.$$

If $\det(I - M\Delta) \ne 0$ for all $\Delta \in \mathbf{D}$ then $\mu_{\mathbf{D}}(M) := 0$.

Elementary property:

- $\mathbf{D} = \{\delta I \ : \ \delta \in C\} \Rightarrow \mu_{\mathbf{D}}(M) = \rho(M)$.
- $\mathbf{D} = C^{n\times n} \Rightarrow \mu_{\mathbf{D}}(M) = \|M\|$.
- In general, $C \cdot I \subset \mathbf{D} \subset C^{n\times n}$ so $\rho(M) \le \mu_{\mathbf{D}}(M) \le \|M\|$.

How good are the bounds?

Let

$$\Delta = \begin{pmatrix} \delta_1 & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

(1) For $M = \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$ with $\beta > 0$ we have

$$\rho(M) = 0, \quad \|M\| = \beta, \quad \mu_{\mathbf{D}}(M) = 0.$$

(2) For $M = \begin{pmatrix} -1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix}$ we have

$$\rho(M) = 0, \quad \|M\| = 1.$$

Since $\det(I - M\Delta) = 1 + (\delta_1 - \delta_2)/2$ we get $\mu_{\mathbf{D}}(M) = 1$.

Thus both bounds are *bad* unless $\rho \approx \bar{\sigma}$.

Invariant transformation

Let us try to find a transformation which does not affect $\mu_{\mathbf{D}}(M)$ but changes ρ and $\bar{\sigma}$.

Define two sets

$$\begin{aligned}
\mathcal{U} &= \{U \in \mathbf{D} \ : \ UU^* = I\}, \\
\mathcal{D} &= \{\text{diag}[D_1, \ldots, D_K, d_1 I_{m_1}, \ldots, d_{L-1} I_{m_{L-1}}, I_{m_L}] \ : \\
&\qquad D_k \in C^{r_k\times r_k}, \ D_k = D_k^* > 0, \ d_l \in R, \ d_l > 0\}.
\end{aligned}$$

Note that for any $\Delta \in \mathbf{D}$, $U \in \mathcal{U}$ and $D \in \mathcal{D}$ it holds

- $U^* \in \mathcal{U}$, $U\Delta \in \mathbf{D}$, $\Delta U \in \mathbf{D}$ (property of the set $\mathbf{D}$).
- $\|U\Delta\| = \|\Delta U\| = \|\Delta\|$ (since $UU^* = I$).
- $D\Delta = \Delta D$ (property of the set $\mathcal{D}$).

Theorem

For all $U \in \mathcal{U}$ and $D \in \mathcal{D}$

1) $\mu_{\mathbf{D}}(M) = \mu_{\mathbf{D}}(UM) = \mu_{\mathbf{D}}(MU)$.

2) $\mu_{\mathbf{D}}(M) = \mu_{\mathbf{D}}(DMD^{-1})$.

Proof: 1) Since for each $U \in \mathcal{U}$

$$\begin{aligned}
\det(I - M\Delta) = 0 \quad &\Leftrightarrow \quad \det(I - MUU^*\Delta) = 0 \\
\Delta \in \mathbf{D} \quad &\Leftrightarrow \quad U^*\Delta \in \mathbf{D}
\end{aligned}$$

we get $\mu_{\mathbf{D}}(M) = \mu_{\mathbf{D}}(MU)$.

2) For all $D \in \mathcal{D}$

$$\det(I - M\Delta) \ = \ \det(I - MD^{-1}\Delta D) = \det(I - DMD^{-1}\Delta)$$

since Δ and D commute. Therefore $\mu_{\mathbf{D}}(M) = \mu_{\mathbf{D}}(DMD^{-1})$.

Improving the bounds

Using Theorem we can tighten the bounds as

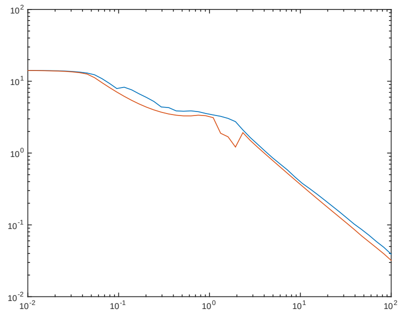
$$\sup_{U\in\mathcal{U}} \rho(UM) \le \mu_{\mathbf{D}}(M) \le \inf_{D\in\mathcal{D}} \|DMD^{-1}\|.$$

Theorem:

$$\sup_{U\in\mathcal{U}} \rho(UM) = \mu_{\mathbf{D}}(M).$$

Theorem: If $2K + L \le 3$ then

$$\mu_{\mathbf{D}}(M) = \inf_{D\in\mathcal{D}} \|DMD^{-1}\|.$$

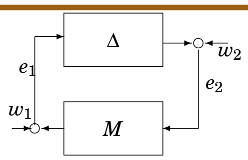



```
>> w=logspace(-2,2);
>> G=ss(randn(5),randn(5),randn(5),0); Gw=frd(G,w);
>> bounds = mussv(Gw,[1 0;-1 0;-1 0;-1 0;-1 0]);
>> loglog(bounds)
```

Remarks:

- In general the quantity $\rho(UM)$ has many local maxima and the local search cannot guarantee to obtain $\mu(M)$.
- Computationally there is a slightly different formulation of the lower bound by Packard and Doyle which gives rise to a power algorithm. It usually works well but has no prove of convergence.
- The upper bound can be computed by convex optimization, but it is not always equal to $\mu(M)$ if $2K + L > 3$.
- It is the upper bound that is the cornerstone of μ synthesis, since it gives a sufficient condition for robust performance.

Structured Robust Stability



Introduce the set

$$\mathcal{T}(\mathbf{D}) = \{\Delta \in RH_\infty \ : \ \Delta(s) \in \mathbf{D} \text{ in RHP}\}.$$

We have the following structured Small Gain Theorem.

Theorem: Let $M \in RH_\infty$. The closed-loop system (M, Δ) is well-posed and internally stable for all $\Delta \in \mathcal{T}(\mathbf{D})$ with $\|\Delta\|_\infty < 1$ if and only if

$$\sup_{\omega\in R} \mu_{\mathbf{D}}(M(j\omega)) \le 1.$$

Proof: The robust stability condition is

$$(I - M\Delta)^{-1} \in RH_\infty, \ \forall \Delta \in \mathcal{T}(\mathbf{D}), \ \|\Delta\|_\infty < 1.$$

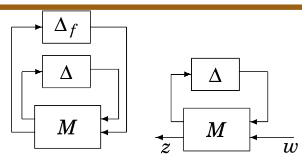
"$\Leftarrow$" It is sufficient to show that

$$\sup_{\text{Re}s\ge 0} \mu_{\mathbf{D}}(M(s)) = \sup_{\omega\in R} \mu_{\mathbf{D}}(M(j\omega)).$$

Obviously $\ge$. The opposite inequality follows from the fact that zeros of $\det(I - M\Delta)$ move continuously with respect to Δ and $\det(I - M\alpha\Delta)$ has no zeros in RHP if $\|M\Delta\|_\infty < 1/\alpha$ (homotopy argument).

"$\Rightarrow$" If $\sup_{\omega\in R} \mu_{\mathbf{D}}(M(j\omega)) > 1$ then by definition of μ there exist $\omega_0 \in R \cup \{+\infty\}$ and Δ_0 with $\|\Delta_0\| < 1$ such that the matrix $I - M(j\omega_0)\Delta_0$ is singular. Next, one can apply the same interpolation argument as in the Small Gain Theorem.

Structured Robust Performance



Let $[p_2, q_2] = size(M_{22})$. Define an augmented block structure

$$\mathbf{D}_P = \begin{pmatrix} \mathbf{D} & 0 \\ 0 & C^{q_2\times p_2} \end{pmatrix}.$$

Theorem: For all $\Delta \in \mathcal{T}(\mathbf{D})$ with $\|\Delta\|_\infty < 1/\beta$ the closed loop is well posed, internally stable and $\|\mathcal{F}_u(M, \Delta)\|_\infty \le \beta$ if and only if

$$\sup_{\omega\in R} \mu_{\mathbf{D}_P}(M(j\omega)) \le \beta.$$

Performance for Constant LFT

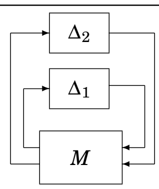
Let $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$ be a complex matrix and suppose that $\mathbf{D}_1$ and $\mathbf{D}_2$ are two defined structures which are compatible in size with M_{11} and M_{22} correspondingly.

Introduce a third structure as

$$\mathbf{D} = \begin{pmatrix} \mathbf{D}_1 & 0 \\ 0 & \mathbf{D}_2 \end{pmatrix}.$$

Theorem:

$$\mu_{\mathbf{D}}(M) < 1 \quad \Leftrightarrow \quad \begin{cases} \mu_{\mathbf{D}_1}(M_{11}) < 1 \\ \sup_{\substack{\Delta_1\in\mathbf{D}_1 \\ \|\Delta_1\|\le 1}} \mu_{\mathbf{D}_2}(\mathcal{F}_u(M, \Delta_1)) < 1 \end{cases}$$



Proof:

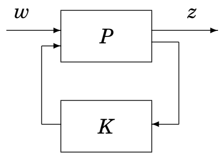
"$\Leftarrow$" Let $\|\Delta_i\| \le 1$. By Schur complement

$$\det(I - M\Delta) = \det \begin{pmatrix} I - M_{11}\Delta_1 & -M_{12}\Delta_2 \\ -M_{21}\Delta_1 & I - M_{22}\Delta_2 \end{pmatrix} =$$
$$= \ \det(I - M_{11}\Delta_1)\det(I - \mathcal{F}_u(M, \Delta_1)\Delta_2) \ne 0.$$

"$\Rightarrow$" Basically the same identity plus (from definition of μ)

$$\mu_{\mathbf{D}}(M) \ge \max\{\mu_{\mathbf{D}_1}(M_{11}), \mu_{\mathbf{D}_2}(M_{22})\}$$

μ Synthesis via $D-K$ Iterations



The problem is to solve

$$\min_{K-\text{stab}} \|\mathcal{F}_l(P,K)\|_\mu.$$

Approximation: $D-K$ iterations for the upper bound

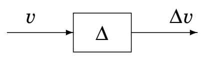
$$\min_{K-\text{stab}} \inf_{D,D^{-1}\in H_\infty} \|D\mathcal{F}_l(P,K)D^{-1}\|_\infty$$

under the condition $D(s)\Delta(s) = \Delta(s)D(s)$.

Remarks:

- Step 1 is the standard H_∞ optimization.
- Step 2 can be reduced to a convex optimization.
- No global convergence is guaranteed.
- Works sometimes in practice.

Integral Quadratic Constraint

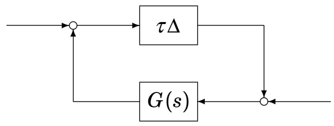


The (possibly nonlinear) operator Δ on $\mathbf{L}_2^m[0,\infty)$ is said to *satisfy the IQC defined by* Π if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} \widehat{v}(i\omega) \\ \widehat{(\Delta v)}(i\omega) \end{bmatrix} d\omega \geq 0$$

for all $v \in \mathbf{L}_2[0,\infty)$.

IQC Stability Theorem



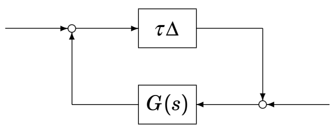
Let $G(s)$ be stable and proper and let Δ be causal.

For all $\tau \in [0,1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQC defined by $\Pi(i\omega)$. If

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \Pi(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

then the feedback system is input/output stable.

IQC Stability Theorem with Several IQCs



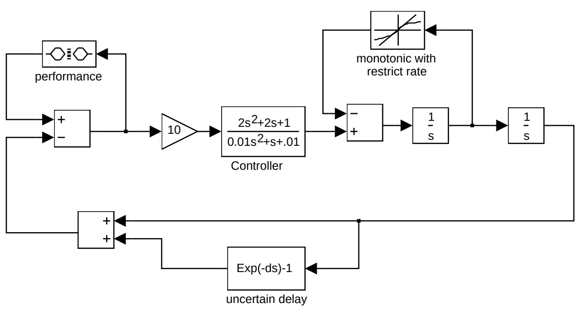
Let $G(s)$ be stable and proper and let Δ be causal. For all $\tau \in [0,1]$, suppose the loop is well posed and $\tau\Delta$ satisfies the IQCs defined by $\Pi_1(i\omega),\ldots,\Pi_m(i\omega)$. If for some $\tau_1,\ldots,\tau_m \geq 0$

$$\begin{bmatrix} G(i\omega) \\ I \end{bmatrix}^* \sum_{k=1}^{m} \tau_k\Pi_k(i\omega) \begin{bmatrix} G(i\omega) \\ I \end{bmatrix} < 0 \quad \text{for } \omega \in [0,\infty]$$

then the feedback system is input/output stable.

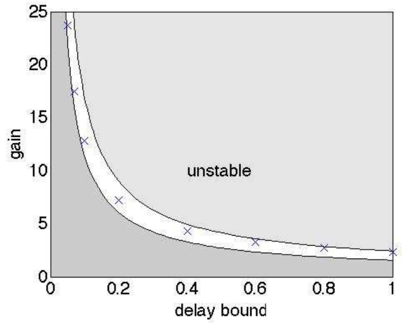
Δ structure	$\Pi(i\omega)$	Condition
Δ passive	$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	
$\|\Delta(i\omega)\| \leq 1$	$\begin{bmatrix} x(i\omega)I & 0 \\ 0 & -x(i\omega)I \end{bmatrix}$	$x(i\omega) \geq 0$
$\delta \in [-1,1]$	$\begin{bmatrix} X(i\omega) & Y(i\omega) \\ Y(i\omega)^* & -X(i\omega) \end{bmatrix}$	$X = X^* \geq 0$ $Y = -Y^*$
$\delta(t) \in [-1,1]$	$\begin{bmatrix} X & Y \\ Y^T & -X \end{bmatrix}$	
$\Delta(s) = e^{-\theta s} - 1$	$\begin{bmatrix} x(i\omega)\rho(\omega)^2 & 0 \\ 0 & -x(i\omega) \end{bmatrix}$	$\rho(\omega) =$ $2\max_{\|\theta\|\leq\theta_0} \sin(\theta\omega/2)$

The IQC toolbox



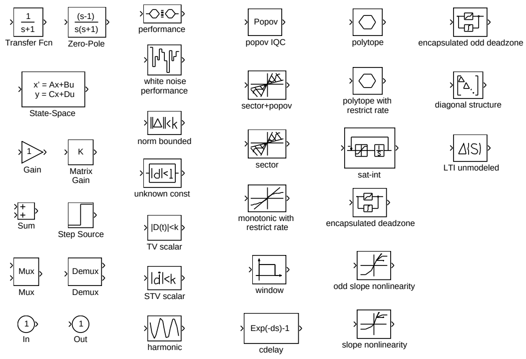

```
>> iqc_gui('fricSYSTEM')

extracting information from fricSYSTEM ...

     scalar inputs:  5
     states:         10
     simple q-forms: 7

Solving with 62 decision variables ...

  ans =     4.7139
```

Verification by IQCs



IQCs prove stability below the lower line.

A library of analysis objects



What have we learned today?

- Pulling out uncertainties gives a diagonal structure
- Structured singular value μ is natural for robust stability but not computable exactly.
- Robust performance equivalent to robust stability with augmented uncertainty.
- $D - K$ iterations as approach to μ synthesis.
- Integral Quadratic Constraints give general framework.