Exercise Session 4

1. Suppose that

$$P(s) = \frac{16 - s}{(s-6)(s+11)}.$$

Assuming a standard feedback configuration, the transfer function from reference to control effort is given by

$$C(s)(I + P(s)C(s))^{-1}.$$

Use `mixsyn` to demonstrate that this plant will always need 'significant control action' to track a reference. Now design a controller with integral action for this process. Try to achieve a control bandwidth of 10 rad/s. Comment on the merits of your design. Are you able to achieve a significantly higher (or lower) control bandwidth?

2. Given a nominal plant $P(s)$, define the multiplicative uncertainty set of size γ to be

$$P_\gamma = \{P_\Delta : P_\Delta = P(I + \Delta), \|\Delta\|_\infty < \gamma\}.$$

Write the problem of maximising γ such that $C(s)$ stabilises every element of P_γ as an $\mathcal{H}_\infty$ optimisation problem that could be solved with `hinfsyn`. Show that given any $\gamma > 0$, if $P(s)$ is stable then there exists a controller that stabilises all $P_\Delta(s) \in P_\gamma$.

3. Most classically motivated design specifications can be written in terms of the sensitivity and complementary sensitivity functions. Given this, when using `mixsyn` we may be tempted to set $W_2 =$ `[]` and solve

$$\inf_{C(s)} \left\| \begin{bmatrix} W_1 S \\ W_3 T \end{bmatrix} \right\|_\infty . \tag{1}$$

We will now try to understand why this might not be a good idea. Suppose that for a given $P(s)$, the optimal solution to eq. 1 is achieved by $C(s)$. Now suppose that the plant is instead given by

$$\bar{P}(s) = \frac{(s+1)^2}{s^2 + \delta s + 1} P(s).$$

Show that if $\delta > 0$, then the controller

$$\bar{C}(s) = \frac{s^2 + \delta s + 1}{(s+1)^2} C_0(s)$$

is optimal with respect to eq. 1. By considering the transfer function

$$\bar{P}(s)(I + \bar{C}(s)\bar{P}(s))^{-1},$$

argue that for small δ this controller will be unsatisfactory. Will including a term with $W_2 \neq 0$ prevent this problem?

4. In the lecture the $\mathcal{H}_\infty$-loopshaping method of Glover and MacFarlane was presented. There it was claimed that if $b_{P,C} > 0.3$, then we will have good robustness guarantees and the loop gain $L = PC$ will approximate the open loop gain of P reasonably well. In this question we will examine this claim in more detail.

(a) Show that if $P(s), C(s)$ are scalar and $b_{P(s),C(s)} \geq \frac{1}{\gamma}$, then

$$\frac{1}{\gamma} - \frac{1}{|P(s)|} \leq \frac{|P(s)C(s)|}{|P(s)|},$$

and for small enough $|P(s)|$

$$\frac{|P(s)C(s)|}{|P(s)|} \leq \frac{\gamma}{1 - |P(s)|\gamma}.$$

In what sense do these bounds show that the loopshape L approximates the gain of P?

(b) Explain the role of the weighting functions in the $\mathcal{H}_\infty$-loopshaping method.

5. In the lecture we saw that if

$$P(s) = \frac{1}{s+1},$$

then we could design a controller which tracked a step response alarmingly well. We will now prove that we can in fact track the step to arbitrary precision! Such a claim also seems a little ridiculous, so we will also show that such a controller has no robustness guarantees to coprime factor uncertainty...

(a) Show that if $Q(s) = C(s)\left(I + P(s)C(s)\right)^{-1}$ is stable, then the controller

$$C(s) = Q(s)(I - P(s)Q(s))^{-1}$$

stabilises $P(s)$.

(b) Use this to argue that

$$\inf_{C(s)} \|W(s)S(s)\|_\infty \iff \inf_{Q(s)\in\mathcal{RH}_\infty} \|W(s)(I - P(s)Q(s))\|_\infty.$$

(c) Consider now

$$Q(s) = \frac{s+1}{s/T+1}.$$

Show that given any $\epsilon > 0$, there exists a $T > 0$ such that

$$\|\frac{1}{s}(I - P(s)Q(s))\|_\infty \leq \epsilon.$$

Why does this imply that we can track a step arbitrarily well? Can you generalise this argument to other transfer functions $P(s)$?

(d) Show that if $T \geq 1$, then the $Q(s)$ from (c) satisfies

$$\|Q(s)\|_\infty = T,$$

and furthermore that this implies that $b_{P,C} \leq \frac{1}{T}$. What does this tell us about the robustness of the of our step tracking controller as $\epsilon \to 0$?