Exercise Session 3

1. In the lecture we saw that if $S = (I + P(s)C(s))^{-1}$ satisifies $\|S(s)\|_\infty \leq 1$, then by the small gain theorem the feedback interconnection of $C(s)$ and $P_\Delta(s)$ is stable for all

$$P_\Delta(s) \in \{P_\Delta(s) : P_\Delta(s) = (I + \Delta(s))^{-1} P(s), \|\Delta(s)\|_\infty < 1\}.$$

Find the analogous uncertainty sets when given a gain bound of 1 on each other element of the gang of four.

2. In the lecture we saw that given a transfer function $G(s)$ with minimal realisation

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right],$$

if there exists a $P \succ 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I \end{bmatrix} \preceq 0,$$

then $\|G(s)\|_\infty \leq 1$. In this question we will prove this assertion. Let $Y(s) = G(s)U(s)$, and let $y(t), u(t)$ be the inverse Laplace transforms of $Y(s), U(s)$. Recall the time domain formula for the $\mathcal{H}_\infty$ norm:

$$\|G(s)\|_\infty = \sup_{u:\|u\|_2 \neq 0} \frac{\|y\|_2}{\|u\|_2}.$$

(i) Show that if $\|y\|_2^2 - \|u\|_2^2 \leq 0$, $\forall u(t)$, then $\|G(s)\|_\infty \leq 1$.

(ii) Define the state space model

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t),\ x(0) = 0, \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

and the function $V(t) = x^T(t)Px(t)$. Show that $0 \geq \dot{V} + y^T y - u^T u$ if and only if

$$\begin{bmatrix} x \\ u \end{bmatrix}^T \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} \leq 0.$$

(iii) Show that for all $T \geq 0$,

$$\int_0^T \dot{V} dt \geq 0.$$

How can we combine this with (i)–(ii) to prove that $\|G\|_\infty \leq 1$? Hint: Consider $\lim_{T\to\infty} \int_0^\infty \dot{V} + y^T y - u^T u dt$.

3. This question is about reducing the conservatism of the small gain theorem using loop transforms. Suppose that Δ is a real number that satisfies $0 < \Delta < 2$. Show that the small gain theorem implies that the negative feedback interconnection of G and Δ is stable if G is stable and

$$|G(j\omega)| \leq \frac{1}{2}.$$

Consider now the transformation

$$\tilde{\Delta} = \frac{1 - \frac{3}{2}\Delta}{1 + \frac{1}{2}\Delta},$$
$$\tilde{G}(s) = \frac{\frac{1}{2} - G(s)}{G(s) + \frac{3}{2}}.$$

Show that $-1 < \tilde{\Delta} < 1$, and that the small gain theorem implies that the feedback interconnection of $\tilde{\Delta}, \tilde{G}$ is stable if $G(s)$ is stable and

$$\mathrm{Re}(G(j\omega)) \geq -\frac{1}{2}.$$

Show that stability of the negative feedback interconnection of G, Δ is equivalent to that of $\tilde{G}, \tilde{\Delta}$. How has this loop transform reduced conservatism?

4. This problem is about proving the converse direction of the small gain theorem. We will prove that (ii)$\Longrightarrow$(i) by showing that if $\|G(s)\|_\infty > 1$, then there exists a $\Delta(s) \in \mathcal{R}^{m\times n}$ satisfying $\|\Delta(s)\|_\infty < 1$ for which

$$(I + G(s)\Delta(s))^{-1}$$

is unstable.

(i) We will first solve the case that Δ is allowed to be a complex matrix. To do this, show that if $\|G(s)\|_\infty > 1$, then there exists a frequency ω_0 and a matrix $\Delta_{\mathbb{C}}$ such that

$$\det(I + G(j\omega_0)\Delta_{\mathbb{C}}) = 0$$

and $\overline{\sigma}(\Delta_{\mathbb{C}}) < 1$.

(ii) We will now try to construct a $\Delta(s) \in \mathcal{R}^{m\times n}$ to interpolate $\Delta_{\mathbb{C}}$ from (i). Suppose that $X \in \mathbb{C}^{n\times n}$ satisfies $X^*X = I$ and $D \in \mathbb{R}^{n\times n}$ satisfies $D^TD = I$. Show that if $(X - D)$ is invertible and scalar, then

$$Q = \left(\frac{s}{\omega_0}\mathrm{Im}\left((X-D)^{-1}\right) + \mathrm{Re}\left((X-D)^{-1}\right)\right)^{-1} + D$$

satisfies $Q\,(j\omega)^*\,Q\,(j\omega) = I$ for all ω and $Q\,(j\omega_0) = X$. Check that the same claims hold in the matrix case numerically (or prove it!). How can we use this construction to find a $\Delta(s) \in \mathcal{R}^{m\times n}$ such that

$$\Delta(j\omega_0) = \Delta_{\mathbb{C}},\ \overline{\sigma}(\Delta(j\omega)) = \overline{\sigma}(\Delta_{\mathbb{C}})? \tag{1}$$

Hint: Consider the SVD of $\Delta_{\mathbb{C}}$.

(iii) The problem with our previous construction is that $\Delta(s)$ is not guaranteed to be stable. We will now show that we remove any such unstable poles without affecting the interpolation requirements. Given $p_i \in \mathbb{C}$, define

$$F_{p_i}(s) = -\frac{(s - p_i)}{(s + p_i^*)}\frac{(s/\omega_0 - \omega_0/p_i)}{(s/\omega_0 + \omega_0/p_i^*)}.$$

Show that $|F_{p_i}(j\omega)| = 1$ for all ω and

$$F_{p_i}(j\omega_0) = \frac{p_i/\omega_0 + \omega_0/p_i}{p_i^*/\omega_0 + \omega_0/p_i^*}.$$

Explain how to combine functions of this form with the construction from (ii) to obtain a stable $\Delta(s) \in \mathcal{R}^{m\times n}$ that meets eq. 1.