Robust Control 2018

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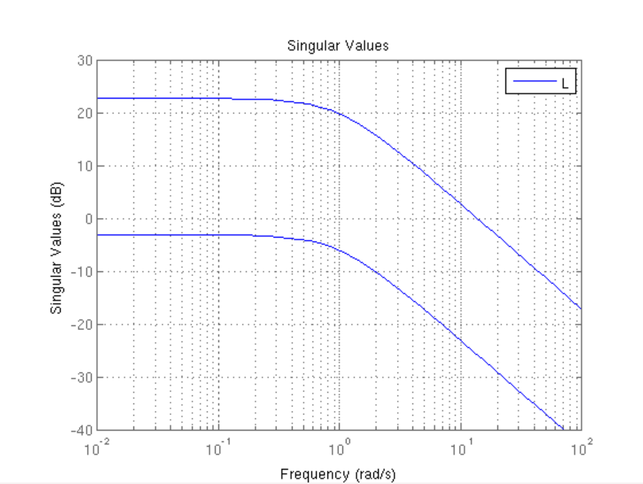
Plan of attack:

Today's topic: Evaluate $\mathcal{H}_\infty$ based robust stability and performance claims.

- $\mathcal{H}_\infty$-norm performance specifications.
- The small gain theorem.
- Robust stability specifications.
- Proof of the small gain theorem:
  - Argument principle.
  - Loop transforms.
  - Instability theorems.

$\mathcal{H}_\infty$ performance specifications

Suppose $L(j\omega)$ is given by the following:



$\mathcal{H}_\infty$ performance specifications

Is $\|\frac{10(s/20+1)}{s+1}S(s)\|_\infty \leq 1$??

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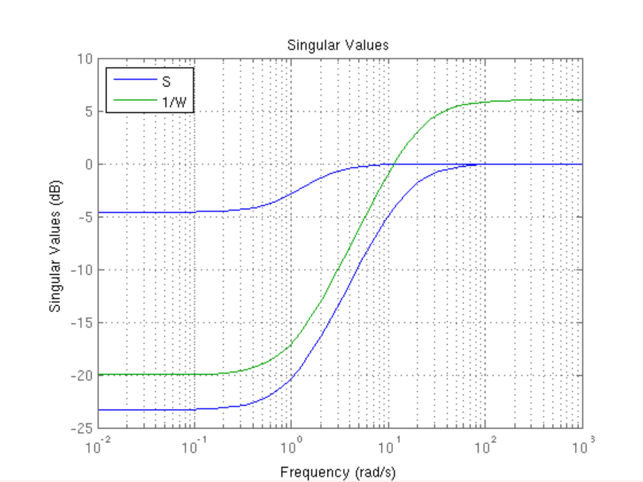

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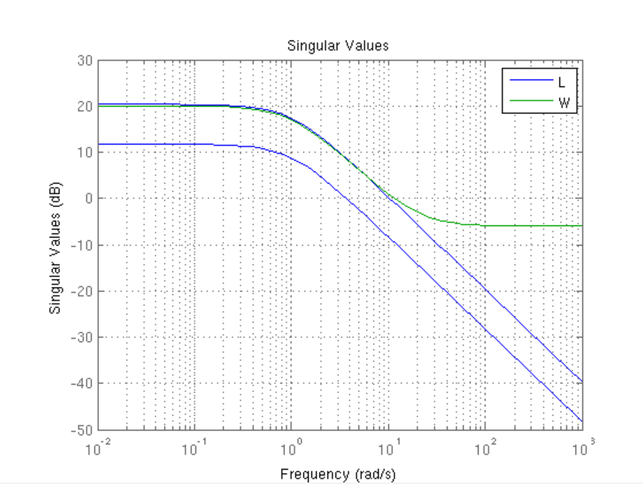

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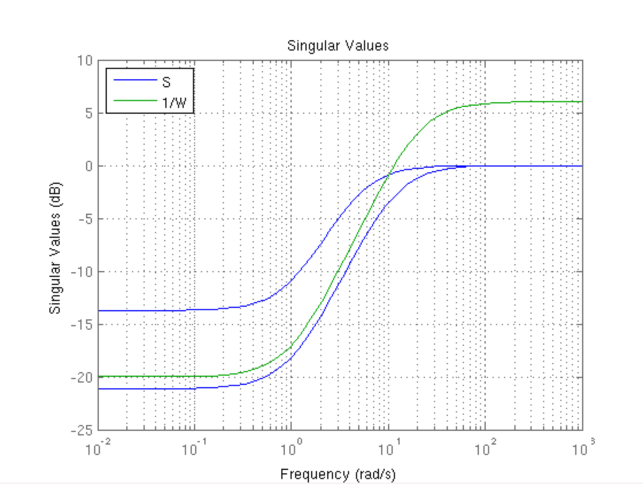

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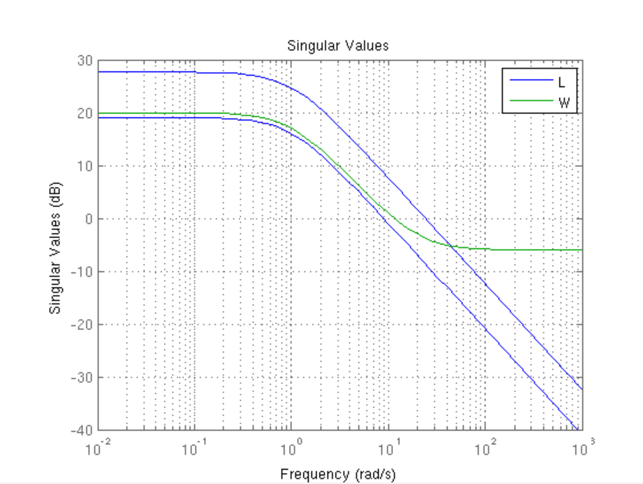
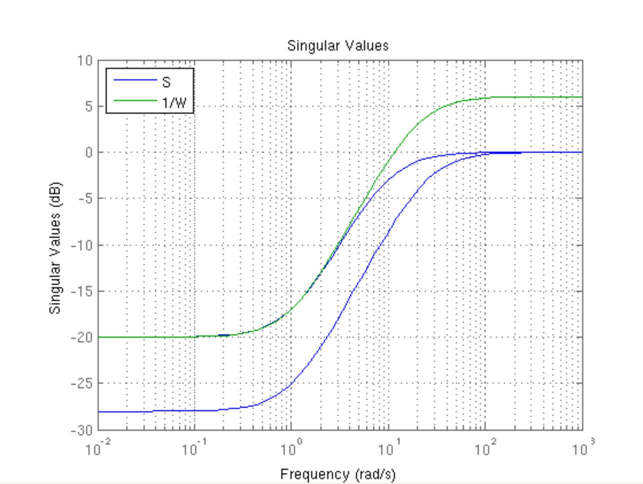


$\mathcal{H}_\infty$ performance specifications



The Bounded real Lemma

To check bounded gain, can use the singular value plot or the bounded real lemma.

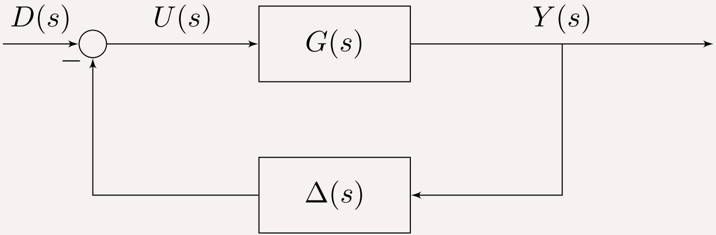
Bounded real Lemma: Given any $G \in \mathcal{R}^{n \times m}$, the following are equivalent:

(i) $\|G\|_\infty \leq 1$.

(ii) For any minimal realisation of G there exists a $P \succ 0$ such that

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + DC^T & D^T D - I \end{bmatrix} \preceq 0.$$

The Small Gain Theorem



Given $G \in \mathcal{R}^{n\times m}$, the following are equivalent:

(i) The feedback interconnection of G and Δ is stable for all $\Delta \in \mathcal{R}^{m\times n}$ such that $\|\Delta\|_\infty < 1$.

(ii) $\|G\|_\infty \leq 1$.

The Small Gain Theorem

Can be generalized significantly. For example:

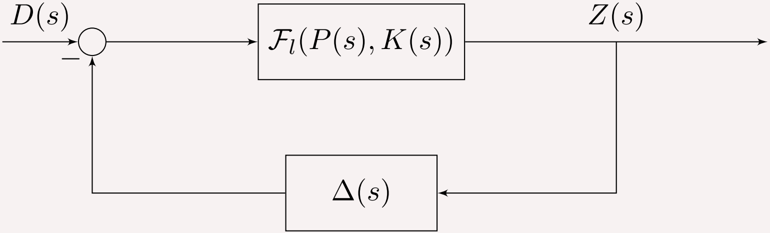
- Real rationality of G and Δ can be removed.
- Linearity of Δ can be removed.

Robust Stability Specifications

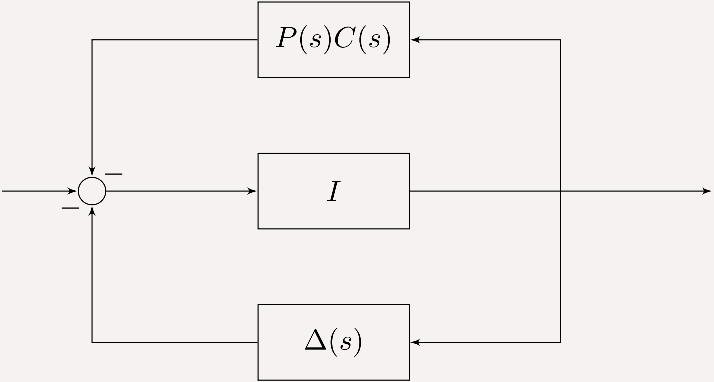
$\mathcal{H}_\infty$-norm performance specifications are robust stability specifications.

Robust Stability Specifications

By the small gain theorem, $\|\mathcal{F}_l(P(s), K(s))\|_\infty \leq 1$ is equivalent to stability of the following for $\|\Delta\|_\infty < 1$:



Robust Stability Specifications



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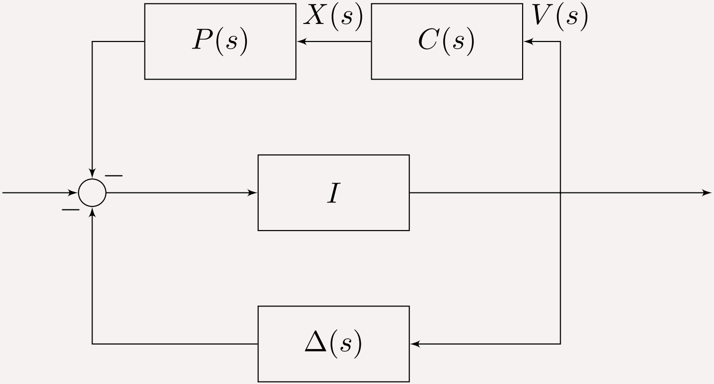

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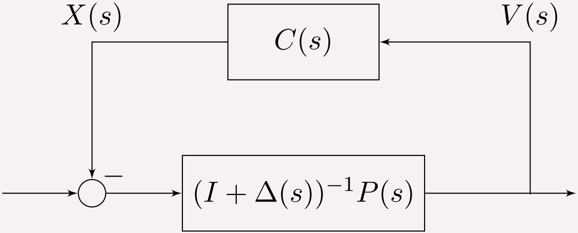

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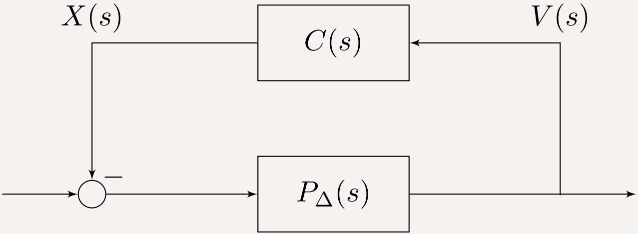

$\|W(s)S(s)\|_\infty \leq 1 \implies$ stability for all

$$P_\Delta(s) \in \{Q(s) : Q(s) = (I + \Delta(s))^{-1}P(s), \|\Delta(s)\|_\infty < 1\}.$$

Robust Stability Specifications

Questions:

1. What are the robust stability equivalents of an $\mathcal{H}_\infty$-norm specification on each element of the gang of four?
2. What about the gang of four stability margin

$$b_{P,C} = \left\| \begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} \begin{bmatrix} -C & I \end{bmatrix} \right\|_\infty^{-1} ?$$

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Exercise

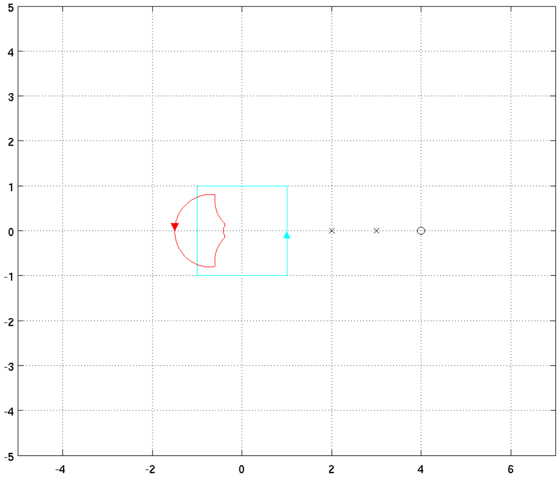
The Argument Principle

Consider

$$g(s) = \frac{s-4}{(s-2)(s-3)}.$$

Let us evaluate $g(s)$ on a square contour.

The Argument Principle



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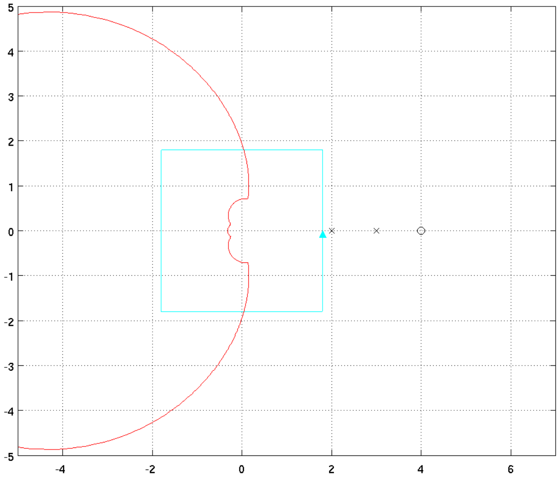

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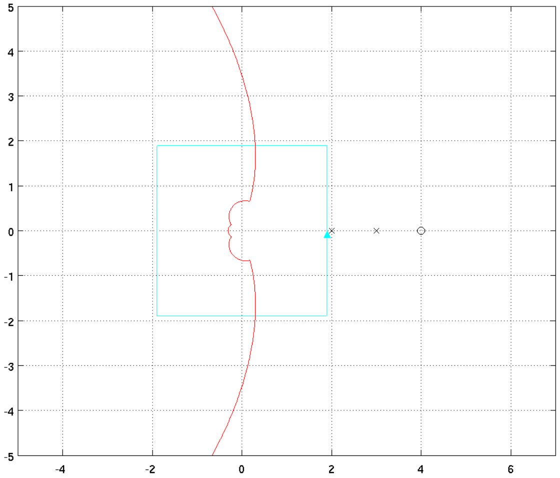

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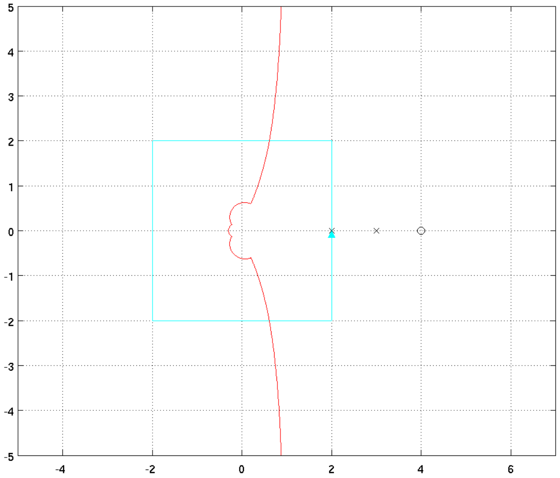

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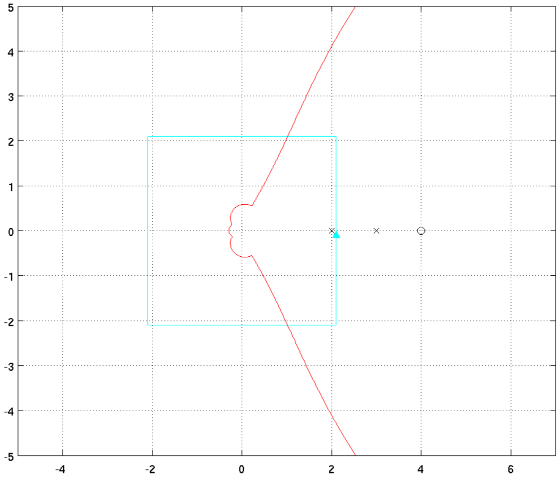

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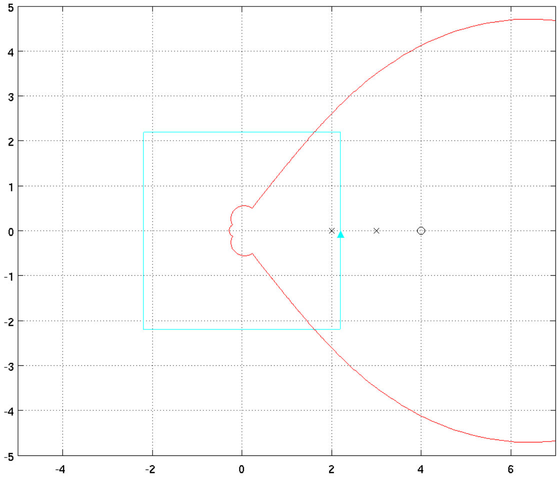

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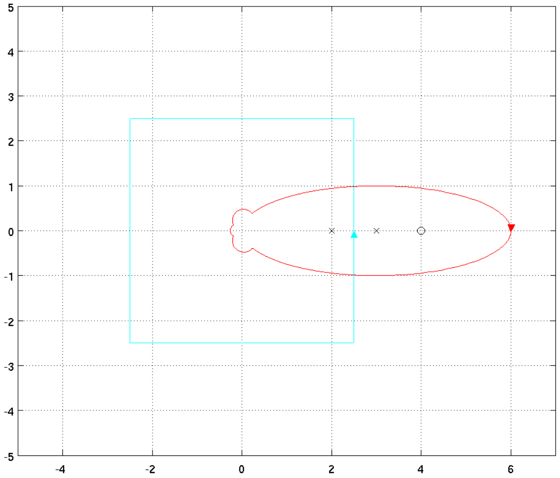

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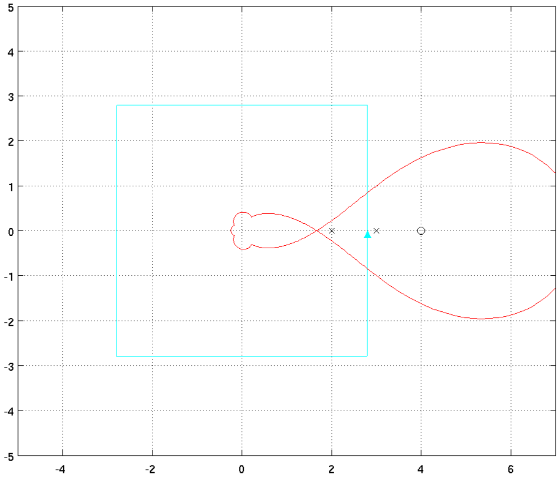

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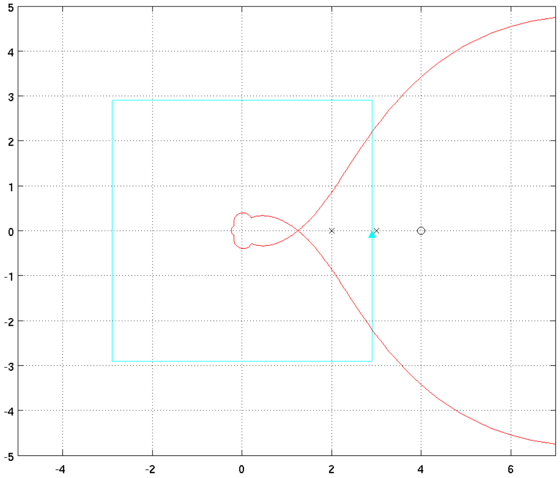

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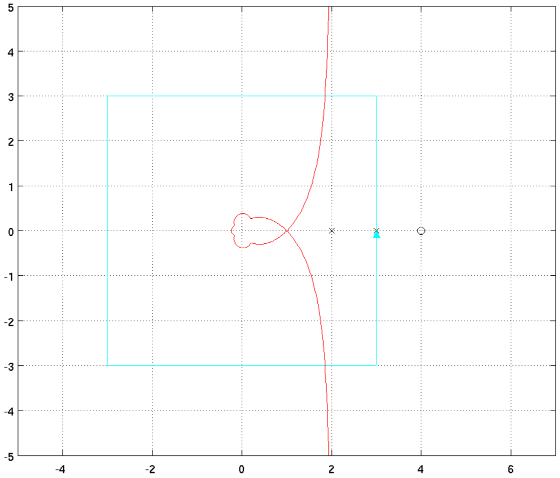

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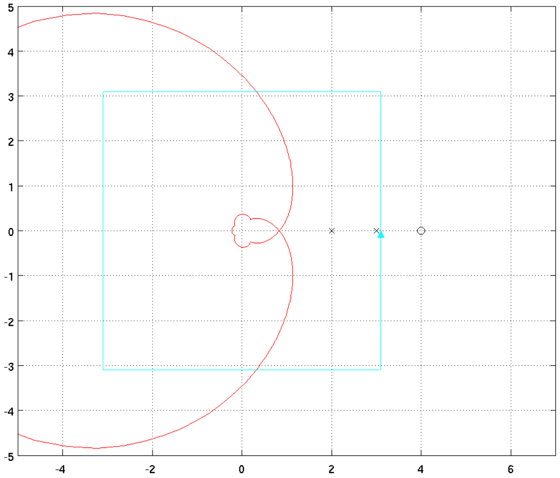

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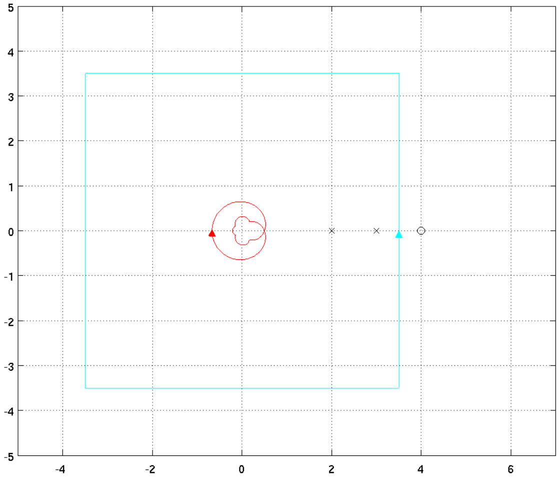

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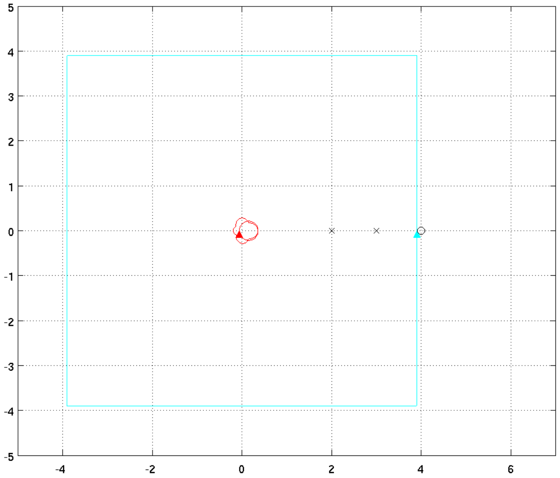

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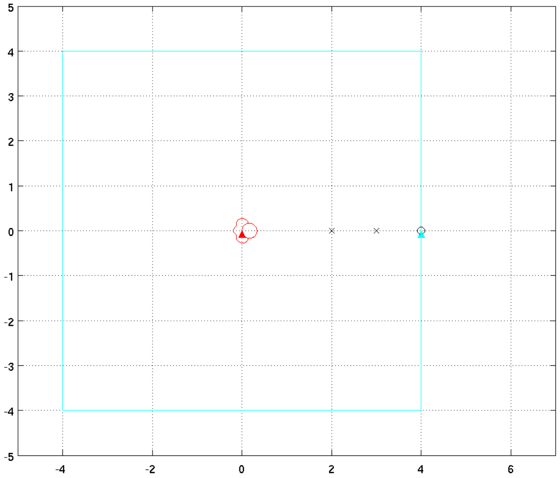

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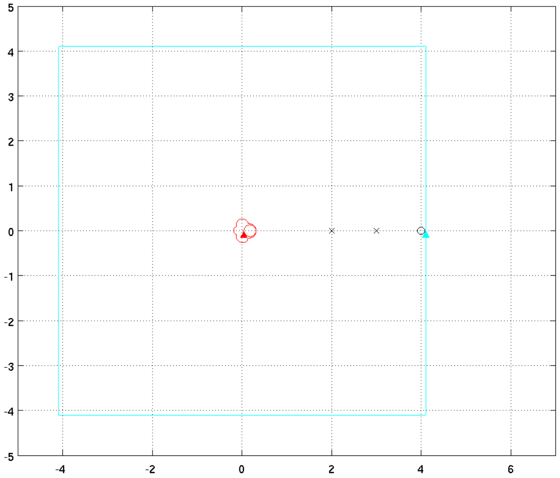

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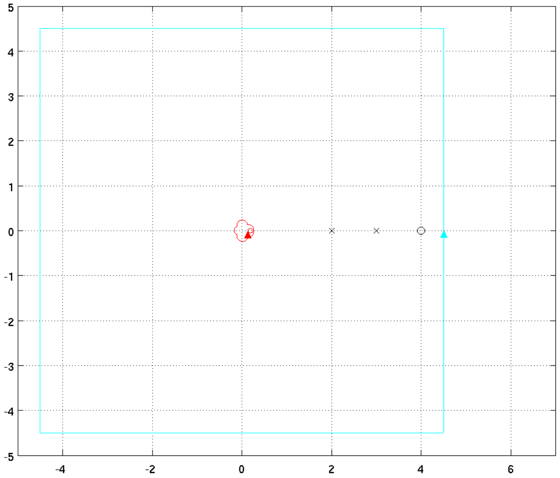


The Argument Principle

Given $g(s) \in \mathcal{R}$ and a closed contour C,

$$\text{w.n.o.} g(s) = Z - P$$

where Z, P are the number of zeros and poles of $g(s)$ contained in C.

The Argument Principle

Given $G(s) \in \mathcal{R}^{n\times n}$ and a closed contour C,

$$\text{w.n.o.}\,\det(G(s)) = Z - P$$

where Z, P are the number of zeros and poles of $G(s)$ contained in C.

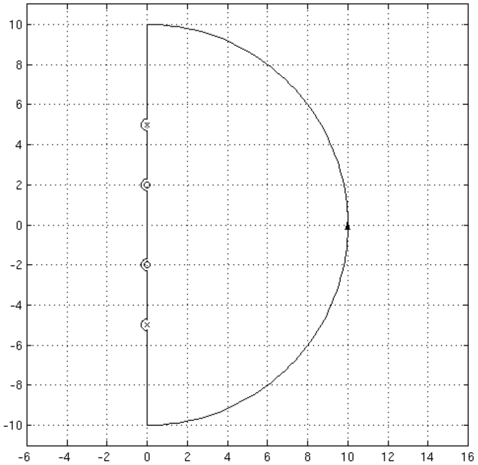
Proof of the Small Gain Theorem (ii)$\Rightarrow$ (i)

We will use the argument principle to show that (ii) implies that

$$\det(I + G(s)\Delta(s)) \neq 0$$

for all s in the closed right half plane.

Proof of the Small Gain Theorem (ii)$\Rightarrow$ (i)



The Nyquist contour.

Proof of the Small Gain Theorem (ii)$\Rightarrow$ (i)

Consider

$$g_\lambda(s) = \det\left(I + \lambda G(s)\Delta(s)\right).$$

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Now vary λ from 0 to 1.

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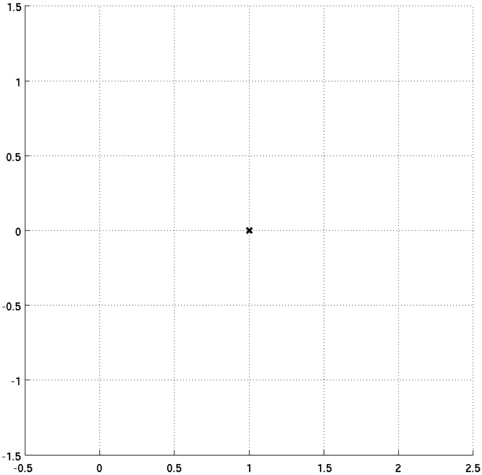

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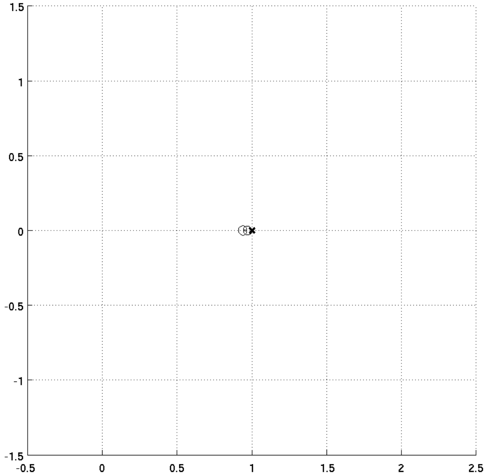

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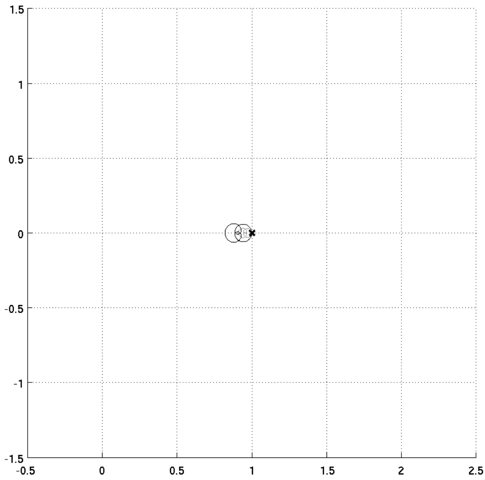

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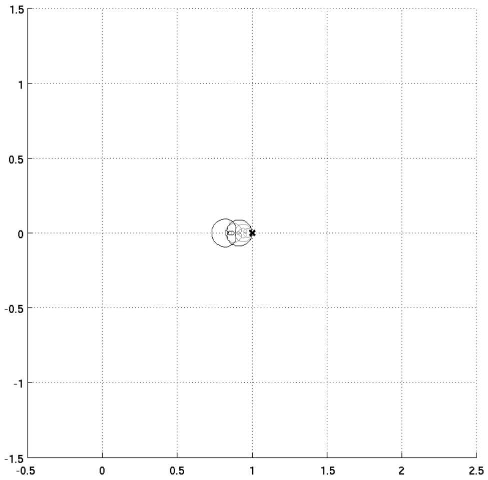

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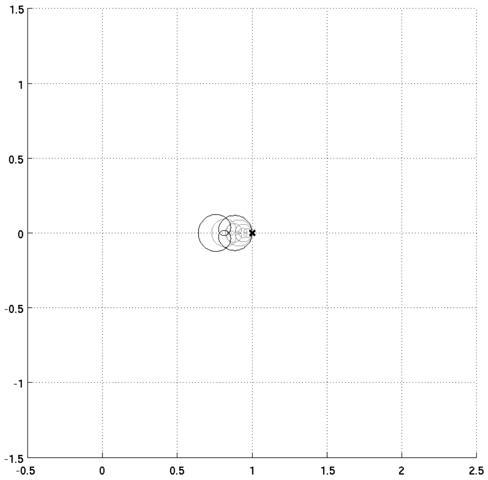

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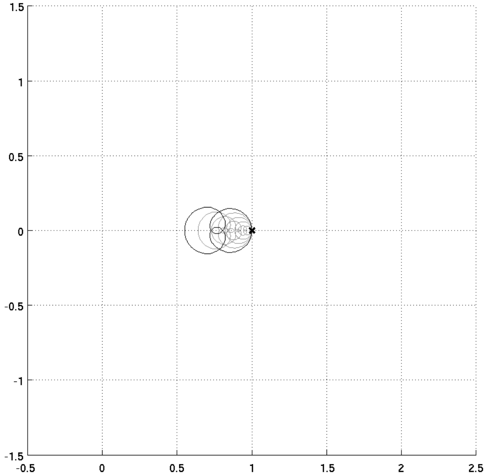

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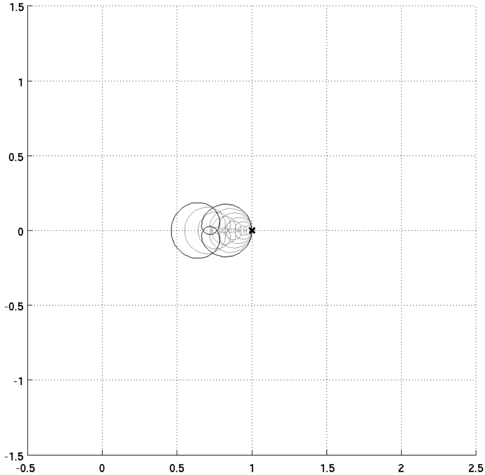

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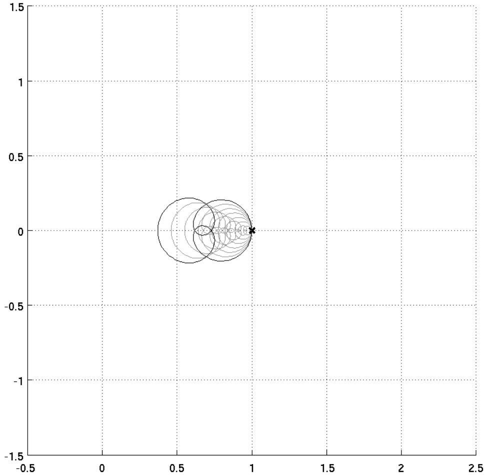

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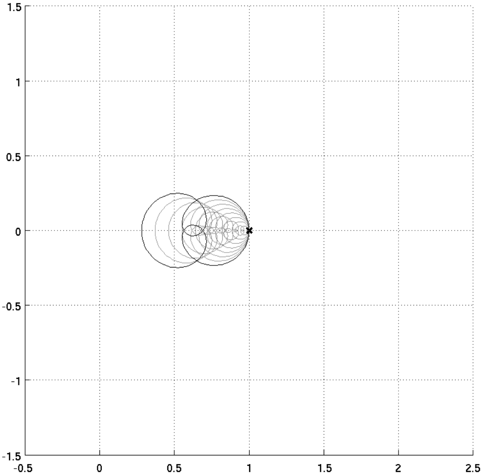

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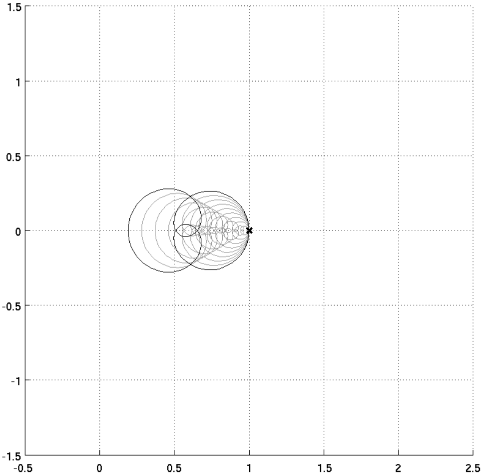

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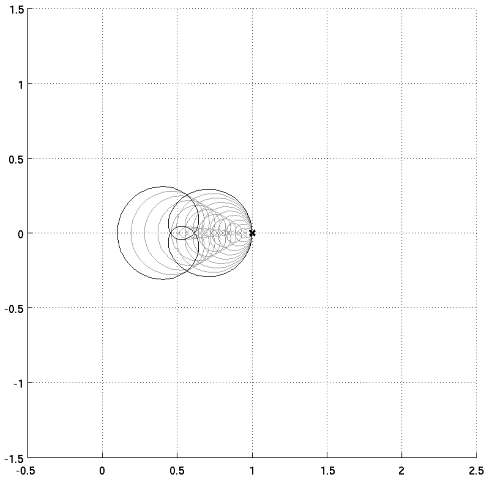

Proof of the Small Gain Theorem (ii)$\Rightarrow$ (i)

Recall that for $C \in \mathbb{C}^{n\times n}$

$$|\det C| = \prod_i \sigma_i(C).$$

Therefore if $\underline{\sigma}(C) > 0$, $\det C \neq 0$.

Proof of the Small Gain Theorem (ii)$\Rightarrow$ (i)

Now recall that $\underline{\sigma}(A+B) \geq \underline{\sigma}(A) - \overline{\sigma}(B)$. Therefore since $\|G\|_\infty \leq 1$ and $\|\Delta\|_\infty < 1$,

$$|g_\lambda(j\omega)| \geq 1 - \lambda\overline{\sigma}(G(j\omega)\Delta(j\omega)) > 0.$$

Proof of the Small Gain Theorem (i)$\Rightarrow$ (ii)

Therefore for every point on the contour, $g_\lambda(s) \neq 0$,

$$\Longrightarrow \text{w.n.o.} g_0(s) = \text{w.n.o.} g_1(s).$$

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By the argument principle, this implies that

$$0 = Z - P.$$

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By the argument principle, this implies that

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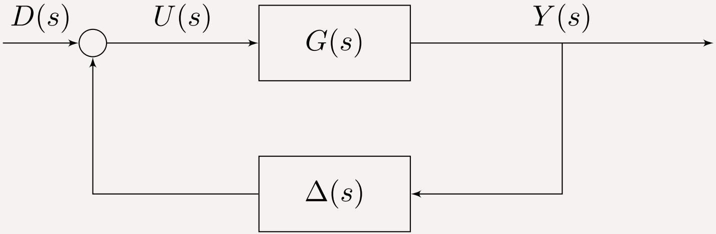
Since $P = 0$, $\implies$ the interconnection is stable.

Proof of the Small Gain Theorem (i)$\Rightarrow$ (ii)

Exercise

Loop Transforms

Can extend the small gain theorem using loop transforms:



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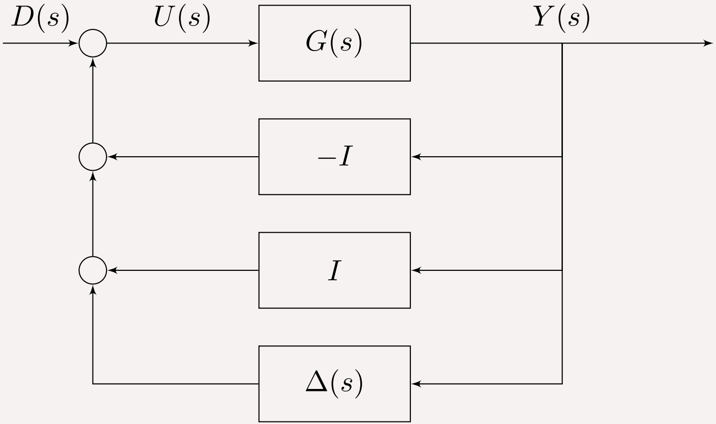

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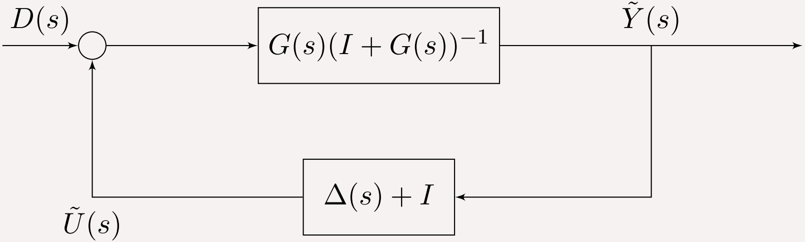

Stable if $\|G(s)(I+G(s))^{-1}\|_\infty \leq 1$ and $\|I+\Delta(s)\|_\infty < 1$.

Loop Transforms

1. Are we sure $(\tilde{Y}(s), \tilde{U}(s))$ bounded implies that $(Y(s), U(s))$ is bounded?
2. How can we interpret such loop transforms?

Using the Projective Line

Recall from last time that we used the projective line to understand $G(s)(I+G(s))^{-1}$. Use the chain description:

$$\begin{bmatrix} I & 0 \\ I & I \end{bmatrix} \begin{bmatrix} G(s) \\ I \end{bmatrix}$$

Using the Projective Line

Signal interpretation:

$$\begin{bmatrix}\tilde{Y}\\ \tilde{U}\end{bmatrix} = \begin{bmatrix}I & 0\\ I & I\end{bmatrix}\begin{bmatrix}Y\\ U\end{bmatrix}, \begin{bmatrix}Y\\ U\end{bmatrix} = \begin{bmatrix}G\\ I\end{bmatrix}U.$$

1. The first equation describes the loop transform.
2. In the second, U generates the 'behaviour' (use coprime factorisation for unstable G)

Using the Projective Line

Let $M \in \mathcal{R}^{n\times n}$ define a loop transform

$$\begin{bmatrix} \tilde{Y}(s) \\ \tilde{U}(s) \end{bmatrix} = M(s) \begin{bmatrix} Y(s) \\ U(s) \end{bmatrix}.$$

Boundedness of $(\tilde{Y}(s), \tilde{U}(s))$ is equivalent to boundedness of $(Y(s), U(s))$ if and only if $M, M^{-1} \in \mathcal{RH}_\infty$.

The Passivity Theorem

Small gain of $Y(s) = G(s)U(s)$ is equivalent to

$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix}^* \begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} \geq 0.$$

The Passivity Theorem

Suppose

$$\begin{bmatrix} Y(s) \\ U(s) \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ -I & I \end{bmatrix} \begin{bmatrix} \tilde{Y}(s) \\ \tilde{U}(s) \end{bmatrix} ?$$

The Passivity Theorem

Small gain is equivalent to:

$$\left(\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} \tilde{Y}(s) \\ \tilde{U}(s) \end{bmatrix}\right)^{*}\begin{bmatrix} -I & 0 \\ 0 & I \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} \tilde{Y}(s) \\ \tilde{U}(s) \end{bmatrix} \geq 0.$$

The Passivity Theorem

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The Passivity Theorem

Small gain is equivalent to:

$$\begin{bmatrix}\tilde{Y}(s)\\ \tilde{U}(s)\end{bmatrix}^* \begin{bmatrix}0 & I\\ I & 0\end{bmatrix} \begin{bmatrix}\tilde{Y}(s)\\ \tilde{U}(s)\end{bmatrix} \geq 0.$$

$$\implies \int_{-\infty}^{\infty} \tilde{y}^T(t)\tilde{u}(t)dt \geq 0.$$

The Passivity Theorem

What about the transfer function given by the chain description:

$$\frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}^{-1}\begin{bmatrix} G(s) \\ I \end{bmatrix}?$$

The Passivity Theorem

What about the transfer function given by the chain description:

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Exercise

The Passivity Theorem

- Small gain of the pair (y, u) is equivalent to passivity of $(\tilde{y}, \tilde{u})$.
- (y, u) are called scattering variables in this context.
- Connections to IQCs, the KYP lemma, chain scattering, ...

Instability Theorems

Observe that in our proof of the small gain theorem we only used our $\mathcal{H}_\infty$-norm bound to bound the largest singular value of G on the Nyquist contour.

Instability Theorems

$\mathcal{L}_\infty$ is the space of essentially bounded measurable functions on the imaginary axis, with norm

$$\|f(j\omega)\|_{\mathcal{L}_\infty} = \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} |f(j\omega)|.$$

Instability Theorems

For real rational functions:

$$\|f(j\omega)\|_{\mathcal{L}_\infty} = \sup_{\omega\in\mathbb{R}} |f(j\omega)|.$$

Instability Theorems

For matrices of real rational functions:

$$\|F(j\omega)\|_{\mathcal{L}_\infty} = \sup_{\omega \in \mathbb{R}} \overline{\sigma}(F(j\omega)).$$

Instability Theorems

By our previous argument, $\|G(j\omega)\|_{\mathcal{L}_\infty} \leq 1$ and $\|\Delta(j\omega)\|_{\mathcal{L}_\infty} < 1$ is sufficient to conclude that

$$\text{w.n.o.}\,\det(I + G\Delta) = 0 = Z - P.$$

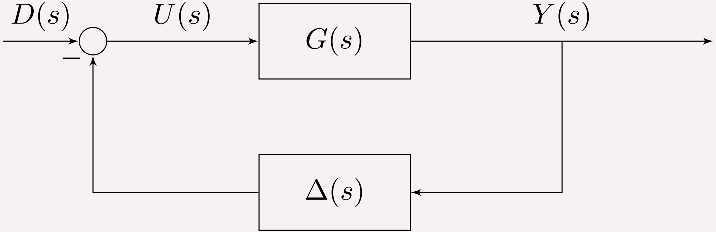
Instability Theorems

By our previous argument, $\|G(j\omega)\|_{\mathcal{L}_\infty} \leq 1$ and $\|\Delta(j\omega)\|_{\mathcal{L}_\infty} < 1$ is sufficient to conclude that

$$\text{w.n.o.}\,\det(I + G\Delta) = 0 = Z - P.$$

$\implies$ that the number of closed loop poles equals the number of open loop poles.

The Small Gain Theorem (2)



Given $G \in \mathcal{R}^{n\times m}$, the following are equivalent:

(i) The feedback interconnection of G and Δ has P unstable poles for all $\Delta \in \mathcal{R}^{m\times n}$ such that $\|\Delta\|_\infty < 1$.

(ii) $\|G\|_{\mathcal{L}_\infty} \leq 1$ and G has P unstable poles.

The Small Gain Theorem (2)

1. Can broaden applicability with the same projective line arguments.
2. Real rational and linear requirements can be relaxed.
3. Can check $\|G\|_{\mathcal{L}_\infty} \leq 1$ using LMIs.