LionSealGrey

Optimal Control 2018

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L1: Functional minimization, Calculus of variations (CV) problem

L2: Constrained CV problems, From CV to optimal control

L3: Maximum principle, Existence of optimal control

L4: Maximum principle (proof)

L5: **Dynamic programming, Hamilton-Jacobi-Bellman equation**

L6: Linear quadratic regulator

L7: Numerical methods for optimal control problems

Exercise sessions (20%):
Solve 50% of problems in advance. Hand-in later.
Mini-project (20%):
Study and present your own optimal control problem.
Written take-home exam (60%).

Summary of L4: Basic problem formulation

Find a control $u \in U \subset \mathbb{R}^m$ that minimizes the cost

$$J(u) = \int_{t_0}^{t_f} \underbrace{L(x(t), u(t))}_{\text{time independent}} dt + K(x_f)$$

where

- $\dot{x} = \underbrace{f(x(t), u(t))}_{\text{time independent}},\ x(t_0) = x_0,\ x \in \mathbb{R}^n, K(x_f) = 0, (t_f, x_f) \in S$
- f, f_x, L, L_x continuous

  - **Basic fixed-endpoint problem (BFEP)** (t_f free, x_f fixed)

    $S = [t_0, \infty) \times \{x_1\}$
  - **Basic variable-endpoint problem (BVEP)** (t_f free, $x_f \in S_1$)

    $S = [t_0, \infty) \times S_1$

    $S_1 = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots h_{n-k}(x) = 0\}$

    $h_i \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R}), i = 1, \ldots, n-k.$

Summary of L4: Maximum principle

Define the Hamiltonian

$$H(x, u, p, p_0) = \langle p, f(x, u)\rangle + p_0 L(x, u).$$

Assume that the basic problem has a solution $(u^*(t), x^*(t))$. Then there exist a function $p^* : [t_0, t_f] \to \mathbb{R}^n$ and a constant $p_0^* \leq 0$ satisfying $(p_0^*, p^*(t)) \neq (0, 0)$ $\forall t \in [t_0, t_f]$ and

1) $\dot{x}^* = H_p(t, x^*, u^*, p^*)$, $\dot{p}^* = -H_x(t, x^*, u^*, p^*)$.

2) $H(x^*(t), u^*(t), p^*(t), p_0^*) \geq H(x^*(t), u(t), p^*(t), p_0^*)$
$\forall t \in [t_0, t_f]$, $\forall u \in U$.

3) $H(x^*(t), u^*(t), p^*(t), p_0^*) = 0 \quad \forall t \in [t_0, t_f]$

4) $\langle p^*(t_f), d\rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1$ (**Only for BVEP**)

$T_{x^*(t_f)} S_1$: tangent space to S_1. *Transversality condition.*

Summary of L4: 6th Step of the Proof

Suppose **Lemma is false.** Then $\exists \hat{y} \in \bar{\mu}$ below $y^*(t^*)$ such that $\hat{y} \in C_{t^*}$ together with a ball $B_\varepsilon \subset C_{t^*}$ $\Rightarrow$ For a suitable $\beta > 0$, one has

$$\hat{y} = y^*(t^*) + \varepsilon\beta\mu$$

Since $B_\varepsilon \subset C_{t^*}$, its points are of the form $y^*(t^*) + \varepsilon\nu$ where $\varepsilon\nu$ are first-order perturbations, arising from the earlier control perturbations.

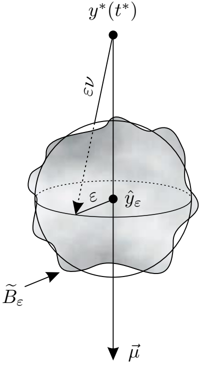

Figure 4.10: Proving Lemma 4.1

- Actual terminal points $y^*(t^*) + \varepsilon\nu + o(\varepsilon)$ of these perturbed trajectories form the set $\tilde{B}_\varepsilon$ which is $o(\varepsilon)$ away from B_ε
- Let $\varepsilon \to 0$, then $\hat{y} := y^*((t^*) + \varepsilon\beta\mu$ approaches $y^*(t^*)$.
- Since the center of B_ε is on $\hat{\mu}$ below $y^*(t^*)$ then for sufficiently small ε, set $\tilde{B}_\varepsilon$ intersects $\bar{\mu}$ below $y^*(t^*)$, too **that contradicts the optimality.**

Summary of L4: Exercise 4.5

Prove that along with the ball B_ε, its warped version $\tilde{B}_\varepsilon$ and $\tilde{\mu}$ must have a nonempty intersection for sufficiently small $\varepsilon > 0$.

The warping map $F(y)$ of the ball B_ε into the warping ball $\tilde{B}_\varepsilon$ is continuous because the terminal points depend continuously on the perturbation parameters, parameterizing the ball B_ε (such as $\omega, a, b, \varepsilon$).

Given an arbitrary $\alpha \in (0, 1)$, the $o(\varepsilon)$ respects $|o(\varepsilon)| < \alpha\varepsilon$ for ε small enough.

For an arbitrary $z \in B_{(1-\alpha)\varepsilon}$ we want to find a point $y \in B_\varepsilon$ such that $F(y) = z$ or what is equivalent, $y = y - F(y) + z$. Actually, the map $G(y) := y - F(y) + z$ has a fixed point because by virtue of $y - F(y) = o(\varepsilon) < \alpha\varepsilon$ and $|z| < (1 - \alpha)\varepsilon$, it maps the ball B_ε to itself.

Outline

1. Motivation: the discrete problem
2. Principle of optimality
3. HJB equation
4. Infinite-horizon problem
5. Sufficient condition for optimality
6. HJB Equation vs. Maximum Principle
7. Nondifferentiable value function: example
8. Viscosity solutions of HJB Equation

Motivating discrete problem

S1:

Discrete system: $x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \ldots, T-1$

$x \in X$ (finite set of N elements)

$u \in U$ (finite set of M elements)

x_T is free

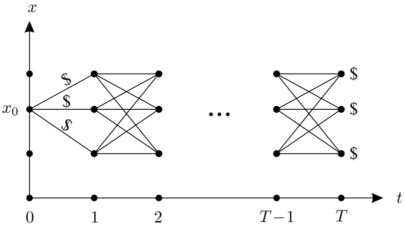

Figure 5.1: Discrete case: going forward

$O(M^T T)$ operations

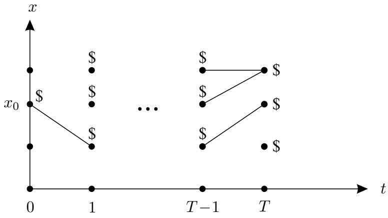

Figure 5.2: Discrete case: going backward

$O(NMT)$ operations

Principle of optimality

State dynamics

$$\dot{x} = f(t, x, u),\ x(t_0) = x_0$$

Fixed-time free-end Bolza problem

$$J(t_0, x_0, u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt + K(x(t_1)) \to \min$$

Family of minimization problems, associated with the cost functional

$$J(t, x, u) = \int_{t}^{t_1} L(s, x(s), u(s))ds + K(x(t_1)),\ t \in [t_0, t_1),\ x \in \mathbb{R}^n$$

Belman's roadmap:

derive a dynamic relationship among these problem by solving all of them!

Principle of optimality (continued)

Value function (optimal cost-to-go)

$$V(t,x) := \inf_{u_{[t,t_1]}} J(t,x,u)$$

where $u_{[t,t_1]}$ is the control restriction to $[t,t_1]$.

Value function boundary condition for Bolza problem

$$V(t_1,x) = K(x) \quad \forall x \in \mathbb{R}^n$$

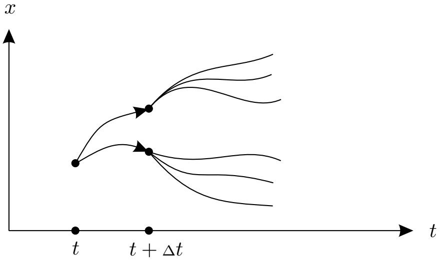

Figure 5.3: Continuous time: principle of optimality

For a general target set $S \subset [t_0,\infty) \times \mathbb{R}^n$ the boundary condition is in the form

$$V(t,x) = K(x) \quad \forall (t,x) \in S$$

Principle of optimality (continued)

$$V(t,x) = \underbrace{\inf_{u_{[t,t+\Delta t]}} \left\{ \int_t^{t+\Delta t} L(s, x(s), u(s))ds + V(t+\Delta t, x(t+\Delta t)) \right\}}_{\bar{V}(t,x)}$$

Proof: Let us show (the reverse inequality is left for your homework)

$$V(t,x) \geq \bar{V}(t,x) \tag{1}$$

By definition $V(t,x) := \inf_{u_{[t,t_1]}} J(t,x,u)$, $\forall \epsilon > 0 \; \exists u_\epsilon$ on $[t, t_1]$:

$$V(t,x) + \epsilon \geq J(t,x,u_\epsilon).$$

Since ϵ is arbitrary, inequality (1) is then verified by virtue of

$$J(t,x,u_\epsilon) = \int_t^{t+\Delta t} L(s, x_\epsilon(s), u_\epsilon(s))ds + J(t+\Delta t, x_\epsilon(t+\Delta t, u_\epsilon)$$

$$\geq \int_t^{t+\Delta t} L(s, x_\epsilon(s), u_\epsilon(s))ds + V(t+\Delta t, x_\epsilon(t+\Delta t) \geq \bar{V}(t,x)$$

Infinitesimal version of the optimality principle

Since

$$x(t+\Delta t) = x + f(t,x,u(t))\Delta t + o(\Delta t)$$

provided that $x(t) = x$, then

$$\int_t^{t+\Delta t} L(s,x(s),u(s))ds = L(t,x,u(t))\Delta t + o(\Delta t)$$

whereas assuming V to be of class C^1 results in

$$V(t+\Delta t, x(t+\Delta t)) = V(t,x)$$
$$+V_t(t,x)\Delta t + < V_x(t,x), f(t,x,u(t))\Delta t) > +o(\Delta t).$$

Plugging the above relations in the optimality principle yields

$$V(t,x) = \inf_{u_{[t,t+\Delta t]}} \{L(t,x,u(t)\Delta t + V(t,x) +$$
$$V_t(t,x)\Delta t + < V_x(t,x), f(t,x,u(t))\Delta t > +o(\Delta t)\},$$

thereby arriving at:

HJB equation

$$\inf_{u_{[t,t+\Delta t]}} \{L(t,x,u(t))\Delta t + V_t(t,x)\Delta t +$$
$$< V_x(t,x), f(t,x,u(t))\Delta t > +o(\Delta t)\} = 0.$$

Being divided by Δt and viewed in the limit as $\Delta t \to 0$, the latter takes the form of the Hamilton-Jacobi-Belman equation

$$-V_t(t,x) = \inf_{u \in U} \{L(t,x,u) + < V_x(t,x), f(t,x,u) >\}$$

to hold true for all $t \in [t_0, t_1)$ and all $x \in \mathbb{R}^n$. Equivalently,

$$V_t(t,x) = \sup_{u \in U} \{-L(t,x,u) - < V_x(t,x), f(t,x,u) >\}$$

HJB equation (continued)

Provided that a global optimal control u^* does exists the infimum/supremum in the HJB equation is replaced by a minimum/maximum and this minimum/maximum is achieved with $u = u^*$:

$$V_t(t,x) = \max_{u \in U}\{-L(t,x^*,u) - < V_x(t,x^*), f(t,x^*,u) >\} = \\ -L(t,x^*,u^*) - < V_x(t,x^*), f(t,x^*,u^*) >$$

In terms of the Hamiltonian

$$H(t,x,u,p) := < p, f(t,x,u) > -L(t,x,u),$$

the latter equality is reproduced in the maximum principle form

$$H(t,x^*(t),u^*(t),-V_x(t,x^*(t))) = \max_{u \in U} H(t,x^*(t),u,-V_x(t,x^*(t)))$$

where the costate vector $p = -V_x(t,x^*(t))$ is explicitly given as the current optimal state function.

Example 1

The standard (scalar) integrator $\dot{x} = u$ **is to be minimized for** a fixed-time t_f, free-endpoint $x(t_f)$ and the cost $L(x,u) = x^4 + u^4$.

The corresponding HJB equation

$$-V_t(t,x) = \inf_{u \in \mathbb{R}} \{x^4 + u^4 + V_x(t,x)u\}, \; V(t_f, x) = 0 \; \forall x \in \mathbb{R}$$

is simplified (by finding the infimum) to

$$-V_t(t,x) = x^4 - 3\left(\frac{1}{4}V_x(t,x)\right)^{\frac{4}{3}}, \; V(t_f, x) = 0 \; \forall x \in \mathbb{R} \qquad (2)$$

Once the HJB equation (2) is solved (what is however hardly possible), the optimal control

$$u^*(t) = -\left(\frac{1}{4}V_x(t, x^*(t))\right)^{\frac{1}{3}}$$

becomes feasible.

Example 2

The minimal time parking problem for

$$\ddot{x} = u, \quad x(t_f) = \dot{x}(t_f) = 0,\ u \in [-1, 1], \quad t_f \to \min$$

The corresponding HJB equation

$$-V_t(t,x) = \inf_{u \in [-1,1]} \{1 + V_{x_1}(t,x)x_2 + V_{x_2}(t,x)u\},\ V(t,0) = 0\ \forall t$$

where the infimum is achieved at

$$u = -sgn\Big(V_{x_2}(t,x)\Big)$$

so that HJB equation is represented as

$$-V_t(t,x) = 1 + V_{x_1}(t,x)x_1 - |V_{x_2}(t,x)|,\ V(t,0) = 0\ \forall t \quad (3)$$

(Further analysis of the HJB equation (3) is among your homework exercises.)

Infinite-horizon problem

The vector field $f = f(x, u)$ **and the cost fuctional** $L = L(x, u)$ **are time-invariant, no terminal cost** $K = 0$**, the final state** $x(t_f)$ **is free, and** $t_f = \infty$

The cost functional becomes $\quad J(u) = \int_{t_0}^{\infty} L(x(t), u(t))dt \to \min$

Just in (autonomous and infinite-horizon) case, the cost functional does not depend on the initial time instant. It follows the value function $V = V(x)$ depends on x only. Thus, the HJB equation reduces to

$$0 = \inf_{u \in U} \{L(x, u)+ < V_x(x), f(x, u) >\} \qquad (4)$$

Particularly, for the **scalar state** $x \in \mathbb{R}$, the HJB equation (4) is **ODE**, and for **Example 1**, it yields

$$x^4 - 3\Big(\frac{1}{4}V_x(t, x)\Big)^{\frac{4}{3}} = 0 \Rightarrow V_x(x) = \Big(\frac{1}{3}\Big)^{\frac{3}{4}} 4x^3 \ \& \ u^*(t) = -\Big(\frac{1}{3}\Big)^{\frac{1}{4}} x^*(t)$$

Sufficient Conditions fro Optimality

All we prove so far is the necessary conditions for optimality

Sufficient condition: Suppose $\hat{V}(t,x) \in C^1 : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the terminal condition $\hat{V}(t_1, x) = K(x)$ and the HJB equation

$$-\hat{V}_t(t,x) = \inf_{u \in U} \{L(t,x,u) + < \hat{V}_x(t,x), f(t,x,u) >\}.$$

Suppose $\hat{u} : [t_0, t_1] \to U$ and the corresponding trajectory $\hat{x} : [t_0, t_1] \to \mathbb{R}^n$, initialized with $x(t_0) = x_0$ satisfies

$$L(t, \hat{x}(t), \hat{u}(t)) + < \hat{V}_x(t, \hat{x}(t)), f(t, \hat{x}(t), \hat{u}(t)) >$$
$$= \min_{u \in U} \{L(t, \hat{x}(t), u) + < \hat{V}_x(t, \hat{x}(t)), f(t, \hat{x}(t), u) >\}$$

(which is representable as the Hamiltonian maximization condition)

Then $\hat{V}(t,x)$ is the optimal cost and $\hat{u}(t)$ is an optimal control.

Proof of the Sufficiency

Specified with $(\hat{u}(t), \hat{x}(t))$, the HJB equation becomes

$$-\hat{V}_t(t, \hat{x}(t)) = L(t, \hat{x}(t), \hat{u}(t)) + < \hat{V}_x(t, \hat{x}(t)), f(t, \hat{x}(t), \hat{u}(t)) > .$$

It follows $0 = L(t, \hat{x}(t), \hat{u}(t)) + \frac{d}{dt}\hat{V}(t, \hat{x}(t))$, thereby yielding

$$0 = \int_{t_0}^{t_1} L(t, \hat{x}(t), \hat{u}(t))dt + \underbrace{\hat{V}(t_1, \hat{x}(t_1))}_{K(\hat{x}(t_1))} - \hat{V}(t_0, \underbrace{\hat{x}(t_0)}_{x_0}).$$

Thus, $\hat{V}(t_0, x_0) = \int_{t_0}^{t_1} L(t, \hat{x}(t), \hat{u}(t))dt + K(\hat{x}(t_1)) = J(t_0, x_0, \hat{u})$.

On the other hand, making the same manipulations for another $x(t) : x(t_0) = x_0$, corresponding to $u(t)$, yields:

$$\hat{V}(t_0, x_0) \leq \int_{t_0}^{t_1} L(t, x(t), u(t))dt + K(\hat{x}(t_1)) = J(t_0, x_0, u).$$

This completes the proof: $J(t_0, x_0, \hat{u}) \leq J(t_0, x_0, u)$.

Historical remarks

The HJB PDE has origins in the work of Hamilton and Jacobi late 1830's. At that time equation served as a necessary optimality condition in the calculus of variations.

Its using as a sufficient optimality condition was proposed by Caratheodori in 1920's

The principle of optimality seems an almost trivial observation dated back to the HJB PDE the work of Bernoulli's solution of the brachistochrone problem in 1697.

Historical remarks (continued)

In early 1950's (slightly before Bellman), the optimality principle was formalized by Isaacs for differential games in terms of the fundamental game theory PDE, bearing his name (also known as Hamilton-Jacobi-Isaacs PDE.

Not clear if Bellman realized connection of his work to Hamilton-Jacobi equation of calculus of variations. This connection was clearly made by Kalman in early 1960's who combined the ideas of Bellman and Caratheodori for derivation of suffcient conditions and who was the first to call the HJB equation.

Pontryagin's maximum principle was being developed independently and in parallel to the work of Bellman and Kalman on dynamic programming.

HJB Equation vs. Maximum Principle (autonomous case)

Canonical state and costate equations $\dot{x}^* = H_p|_*,\ \dot{p}^* = -H_x|_*$ (5)

Maximum principle $u^*(t) = \arg\max_{u \in U} H((x^*(t), u, p^*(t))$ (6)

HJB equation yields $u^*(t) = \arg\max_{u \in U} H((x^*(t), u, -V_x(t, x^*(t)))$ (7)

Is the maximum principle (6) deducible from HJB-based relation (7)?

It happens if $p^*(t) = -V_x(t, x^*(t))$ where the value function V reads

$$-V_t(t, x^*(t)) = L(t, x^*(t), u^*(t)) + < V_x(t, x^*(t)), f(t, x^*(t), u^*(t)) > .$$

Since $V(t_1, x) = K(x)$ it does match the boundary condition $p^*(t_1) = -K_x(x^*(t_1))$ of the maximum principle.

Thus, it remains to establish that $p^*(t) = -V_x(t, x^*(t))$ satisfies the costate equation (5) (homework).

Example: nondifferentiable value function

So, the maximum principle is actually deducible from the Bellman'a dynamic programming provided that $V(t,x)$ is at least of class C^1.

Is in general the value function smooth?

Fixed-time free-endpoint scalar optimal control problem

$$\dot{x} = xu, \quad J(u) = x(t_1) \to \min_{u \in [-1,1]}$$

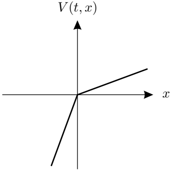

Figure 5.4: Value function nondifferentiable at $x = 0$

$$V(t,x) = \begin{cases} e^{-(t_1-t)}x & \text{if} \quad x > 0 \\ e^{t_1-t}x & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$$

HJB equation

$$-V_t = \inf_u \{V_x u\} = -|V_x x|, \quad V(t_1, x) = x$$

does not admit a C^1 solution (what is typical for constrained control).

Introduction to HJB nonsmooh solutions

One-sided differentials Let $v(x) \in C^0 : \mathbb{R}^n \to \mathbb{R}$. Vector $\xi \in \mathbb{R}$ is a *super-differential* $D^+v(x)$ of v at x iff $\forall y$ near x it reads

$$v(y) \leq v(x) + < \xi, (y - x) > + o(|y - v|).$$

Similarly, $\xi \in \mathbb{R}$ is a *sub-differential* $D^-v(x)$ iff $\forall y$ near x it reads

$$v(y) \geq v(x) + < \xi, (y - x) > - o(|y - v|)$$

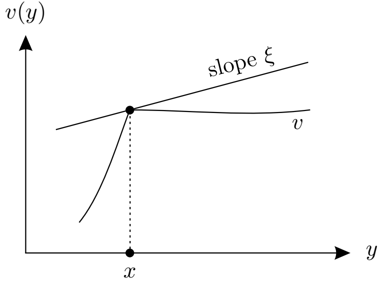

Figure 5.5: (a) super-differential, (b) sub-differential

Example: sub(super)-differentials

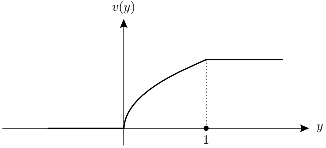

Figure 5.6: The function in Example 5.3

$$v(x) = \begin{cases} 0 & \text{if} \quad x > 0 \\ \sqrt{x} & \text{if} \quad 0 \leq x \leq 1 \\ 1 & \text{if} \quad x > 1 \end{cases}$$

The one-sided differentials

$$D^+v(0) = \emptyset, \quad D^-v(0) = [0, \infty)$$

$$D^+v(1) = [0, \frac{1}{2}], \quad D^-v(1) = \emptyset$$

Test functions

Super(sub)-differential criterion

A vector $\xi \in D^+v(x)$ if and only if $\exists$ a test function $\phi \in C^1 : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\phi(x) = \xi,\ \phi(x) = v(x)$, and $\phi(y) \geq v(y)\ \forall y$ near x, i.e, $\phi - v$ has a local minimum at x.

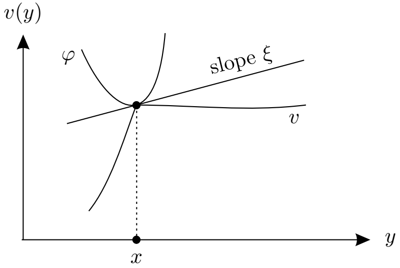

Figure 5.7: Characterization of a super-differential via a test function

Similarly, $\xi \in D^-v(x)$ if and only if $\exists$ a test function $\phi \in C^1 : \mathbb{R}^n \to \mathbb{R}$ such that $\nabla\phi(x) = \xi$ and $\phi - v$ has a local maximum at x.

Relations with classical differentials

If v is differentiable at x, then

$$D^+v(x) = D^-v(x) = \{\nabla v(x)\}$$

If $D^+v(x)$ and $D^-v(x)$ are both nonempty, then v s differentiable at x and the above relation holds.

Non-emptness and denseness The sets $\{x : D^+v(x) \neq \emptyset\}$ and $\{x : D^-v(x) \neq \emptyset\}$ are both non-empty, and dense in the domain of v.

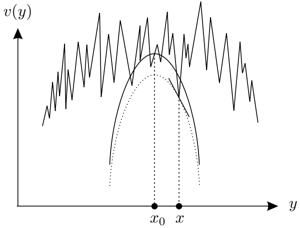

Figure 5.8: Proving denseness

Viscosity solutions of PDEs

$$F(x, v(x), \nabla v(x)) = 0 \tag{8}$$

A viscosity subsolution of (9) with a continuous left-hand side is a continuous function $v : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(x, v(x), \xi) \leq 0 \quad \forall \xi \in D^+(v(x)),\ \forall x$$

This is equivalently to say that $\forall x$ one has $F(x, v(x), \xi) \leq 0 \, \forall \, C^1$-test functions $\phi(x)$ such that $\phi - v$ has a local minimum at x.

A viscosity subsolution of (9) is a continuous function $v : \mathbb{R}^n \to \mathbb{R}$ such that

$$F(x, v(x), \xi) \geq 0 \quad \forall \xi \in D^-(v(x)),\ \forall x$$

This is equivalently to say that $\forall x$ one has $F(x, v(x), \xi) \geq 0 \, \forall \, C^1$-test functions $\phi(x)$ such that $\phi - v$ has a local maximum at x.

Finally, v is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution

Example: viscosity solution

Consider the scalar PDE

$$1 - |\nabla v(x)| = 0$$

with $F(x, v, \xi) = 1 - |\xi|$. By inspection, the functions $v(x) = x$ and $v(x) = -x$ are both classical solutions of the above PDE (as they satisfy the PDE outside the origin i.e., almost everywhere).

The function $v(x) = |x|$ is a viscosity solution. Indeed,for checking the PDE at $v = 0$, note first that $D^+v(0) = \emptyset$ hence $F(x, v(x), \xi) \leq 0$ is satisfied trivially. Second, $D^-v(0) = [-1.1]$ and $1 - |\xi| \geq 0$ holds $\forall\, \xi \in [-1, 1]$.

Lack of sign symmetry of viscosity solution

By inspection, $v(x) = |x|$ is not a viscosity solution of $|\nabla v(x)| - 1 = 0$

About terminology

$$F(x, v_\epsilon(x), \nabla v(x)) = \epsilon \Delta v_\epsilon(x) \quad \textbf{viscous fluid equation} \qquad (9)$$

HJB equation and the value function

$$-V_t(t,x) - \inf_{u \in U}\{L(t,x,u) + < V_x(t,x), f(t,x,u) >\} = 0. \quad (10)$$

Main result for a fixed-time free-end point optimal control

The value function V is a unique viscosity solution of the HJB equation (10) with the boundary condition $V(t_1, x) = K(x), \ \forall x \in \mathbb{R}^n$.

Why V a viscosity solution of (10) with the correct sign convention

Given an arbitrary (t_0, x_0), one needs to make sure that $\forall \ C^1$-test function $\phi(t,x)$ such that $\phi - V$ attains a local minimum at (t_0, x_0), the inequality holds (proving the claim for viscosity subsolution is left for yourself):

$$\phi_t(t_0,x_0) - \inf_{u \in U}\{L(t_0,x_0,u) + < \phi_x(t_0,x_0), f(t_0,x_0,u) >\} \leq 0.$$

Proof of the value function to be a viscosity solution

Suppose on the contrary, $\exists$ C^1-function ϕ and a control value $u_0 \in U$ such that

$$\phi(t_0, x_0) = V(t_0, x_0),\ \phi(t, x) \geq V(t, x)\ \forall\ (t, x) \text{ near } x \quad (11)$$

$$\phi_t(t_0, x_0) - L(t_0, x_0, u_0) - \Big\langle \phi_x(t_0, x_0), f(t_0, x_0, u_0) \Big\rangle > 0 \quad (12)$$

Consider the state trajectory, initialized with $x(t_0) = x_0$ and resulting from applying $u = u_0$ on $[t_0, t_0 + \Delta t]$. It follows

$$\begin{aligned} V(t_0 + \Delta t, x(t_0 + \Delta t) - V(t_0, x_0) &\leq \phi(t_0 + \Delta t, x(t_0 + \Delta t) - \phi(t_0, x_0) \\ &= \int_{t_0}^{t_0+\Delta t} \frac{d}{dt}\phi(t, x(t))dt = \int_{t_0}^{t_0+\Delta t} \Big(\phi_t(t, x(t)) \\ + < \phi_x(t, x(t)), f(t, x(t), u_0) > \Big)dt &< -\int_{t_0}^{t_0+\Delta t} L(t, x(t), u_0)dt \end{aligned}$$

Proof (continued)

Thus

$$V(t_0, x_0) > \int_{t_0}^{t_0+\Delta t} L(t, x(t), u_0)dt + V(t_0 + \Delta t, x(t_0 + \Delta t)) \quad (13)$$

that **contradicts to the principle of optimality:**

$$V(t_0, x_0) \leq \int_{t_0}^{t_0+\Delta t} L(t, x(t), u_0)dt + V(t_0 + \Delta t, x(t_0 + \Delta t)).$$

Relation (13) implies that the optimal cost-to-go is higher than the cost of applying the constant control $u = u_0$ on $[t_0, t_0 + \Delta t]$ followed by an optimal control on the remaining interval that cannot be true.