Optimal Control 2018

Kaoru Yamamoto

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L1: Functional minimization, Calculus of variations (CV) problem
L2: Constrained CV problems, From CV to optimal control
L3: **Maximum principle, Existence of optimal control**
L4: Maximum principle (proof)
L5: Dynamic programming, Hamilton-Jacobi-Bellman equation
L6: Linear quadratic regulator
L7: Numerical methods for optimal control problems

Exercise sessions (20%):
Solve 50% of problems in advance. Hand-in later.
Mini-project (20%):
Study and present your own optimal control problem.
Written take-home exam (60%).

Summary of Lecture 2

- Calculus of variations problems
  - Integral, non-integral constraints, Lagrange multipliers
  - Piecewise $\mathcal{C}^1$ curves, corner points, necessary conditions for strong extrema
- Optimal control via calculus of variations
  - The first variation and the Hamiltonian
  - Conjectured necessary conditions for optimality (Hamiltonian maximization)

Limitations in the variational approach

- $U = \mathbb{R}^m$ guarantees u^* to be an interior point. What if U has a boundary and $u^* \in \partial U$? The Hamiltonian still takes a maximum at $u^*(t)$ but cannot be established by variational approach.
- $S = \{t_1\} \times \{x_1\}$ instead of $S = \{t_1\} \times \mathbb{R}^n \Rightarrow$ Admissible ξ changes and $\delta J(u^*, \xi) = -\int_{t_0}^{t_1} \langle H_u(t, x^*, u^*, p^*), \xi \rangle dt = 0$ is no longer strong enough to conclude $H_u(t, x^*, u^*, p^*) \equiv 0$.
- Differentiability of H w.r.t. u was assumed $\Rightarrow$ differentiability of f and L is assumed. e.g., $J(u) = \int_{t_0}^{t_1} |u(t)| dt$ not allowed.
- Only small deviation in both x and u allowed. Some reasonable control laws left out.

Outline

- **Maximum principle** for
  - basic fixed-endpoint control problem
  - basic free-endpoint control problem
- Other types of problems by change of variables
- Time-optimal control problems and related problems
  - linear systems $\Rightarrow$ often bang-bang principle
  - Minimun time-fuel control and bang-off-bang principle
  - Fuller's problem and Zeno behavior
- A sparsity property of L^0- and L^1- optimal control
  - Maximum hands-off control
- Existence of optimal control
  - necessary conditions could be misleading if no solution exists.

Basic problem formulation

Find a control $u \in U \subset \mathbb{R}^m$ that minimizes the cost

$$J(u) = \int_{t_0}^{t_f} \underbrace{L(x(t), u(t))}_{\text{time independent}} dt + K(x_f)$$

where

- $\dot{x} = \underbrace{f(x(t), u(t))}_{\text{time independent}}, \; x(t_0) = x_0, \; x \in \mathbb{R}^n, K(x_f) \equiv 0, (t_f, x_f) \in S$
- f, f_x, L, L_x continuous

- **Basic fixed-endpoint problem (BFEP)** (t_f free, x_f fixed)
  $S = [t_0, \infty) \times \{x_1\}$
- **Basic variable-endpoint problem (BVEP)** (t_f free, $x_f \in S_1$)
  $S = [t_0, \infty) \times S_1$
  $S_1 = \{x \in \mathbb{R}^n : h_1(x) = h_2(x) = \cdots h_{n-k}(x) = 0\}$
  $h_i \in \mathcal{C}^1(\mathbb{R}^n \to \mathbb{R}), i = 1, \dots, n-k.$

Maximum principle

Define the Hamiltonian

$$H(x, u, p, p_0) = \langle p, f(x, u) \rangle + p_0 L(x, u).$$

Assume that the basic problem has a solution $(u^*(t), x^*(t))$. Then there exist a function $p^* : [t_0, t_f] \to \mathbb{R}^n$ and a constant $p_0^* \le 0$ satisfying $(p_0^*, p^*(t)) \ne (0, 0) \; \forall t \in [t_0, t_f]$ and

1) $\dot{x}^* = H_p(t, x^*, u^*, p^*), \; \dot{p}^* = -H_x(t, x^*, u^*, p^*).$

2) $H(x^*(t), u^*(t), p^*(t), p_0^*) \ge H(x^*(t), u(t), p^*(t), p_0^*)$
$\forall t \in [t_0, t_f], \; \forall u \in U.$

3) $H(x^*(t), u^*(t), p^*(t), p_0^*) = 0 \quad \forall t \in [t_0, t_f]$

4) $\langle p^*(t_f), d \rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1$ (Only for BVEP.)
$T_{x^*(t_f)} S_1$: tangent space to S_1. *Transversality condition.*

Transversality condition

$$\langle p^*(t_f), d \rangle = 0 \quad \forall d \in T_{x^*(t_f)} S_1. \tag{1}$$

$$T_{x^*(t_f)} S_1 = \{d \in \mathbb{R}^n : \langle \nabla h_i(x^*(t_f)), d \rangle = 0, i = 1, \dots n-k\}$$

- (1) means $p^*(t_f)$ is a linear combination of $\nabla h_i(x^*(t_f))$.
- $S_1 = \{x_1\} \implies$ (1) is true for all $p^*(t_f)$.
- $S_1 = \mathbb{R}^n$ (i.e., $k = n$) $\implies p^*(t_f) = 0$.
- In general, k degrees of freedom for $x^*(t_f)$ and $n-k$ degrees of freedom for $p^*(t_f)$.

Remarks

- The maximum principle gives **necessary** conditions.
- It gives all possible optimal control candidates.
- An optimal control may not even exist!
  (It does exist, in fact, for many problems of interest.)
- p_0^*: abnormal multiplier.
  - $p_0^* = 0$: abnormal case (L does not matter.)
  - $p_0^* \neq 0 \Rightarrow (p_0^*, p^*(t))$ can be normalized so that $p_0^* = -1$.

Cases not in the basic setting

- Fixed terminal time
- Time-dependent system and cost
- Terminal cost
- Initial sets

Changes of variables can make them fit into our framework.

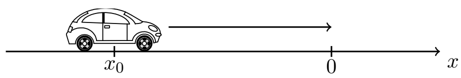
Time-optimal problems

Minimize $\quad J(u) = t_f - t_0 = \int_{t_0}^{t_f} 1\, dt$

With control constraints $|u_i(t)| \leq u_i^{\max}$

- often bang-bang control as a solution for linear systems.

Example: double integrator



$$\ddot{x}_1 = u, \quad u \in [-1, 1].$$

- $L \equiv 1$, $x := (x_1, x_2)^T$, $\dot{x} = (x_2, u)^T \Rightarrow H = p_1 x_2 + p_2 u + p_0$.
- The adjoint equation

$$\begin{pmatrix} \dot{p}_1^* \\ \dot{p}_2^* \end{pmatrix} = \begin{pmatrix} -H_{x_1}|_* \\ -H_{x_2}|_* \end{pmatrix} = \begin{pmatrix} 0 \\ -p_1^* \end{pmatrix} \Rightarrow \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 t + c_2 \end{pmatrix}.$$

- The Hamiltonian maximization.

What is u^*?

Example: double integrator cont.

- $L \equiv 1$, $x := (x_1, x_2)^T$, $\dot{x} = (x_2, u)^T \Rightarrow H = p_1 x_2 + p_2 u + p_0$.
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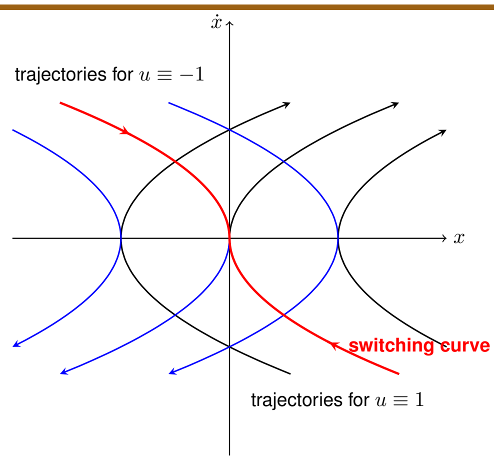
- The Hamiltonian maximization.

$$u^*(t) = \operatorname{sgn}(p_2^*(t)) = \begin{cases} 1 & \text{if} \quad p_2^*(t) > 0, \\ -1 & \text{if} \quad p_2^*(t) < 0, \\ ? & \text{if} \quad p_2^*(t) = 0. \end{cases}$$

Note that $p_2^*(t) \equiv 0 \Rightarrow p_1^*(t) \equiv 0 \Rightarrow H|_* = p_0^* = 0.$

$\therefore p_2^*(t) \not\equiv 0$ (nontriviality condition).

Bang-bang time-optimal control of the double integrator



Bang-bang principle for linear systems

Consider a system with general linear time-invariant dynamics

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^m$ and $u_i \in [-1, 1], i = 1, \ldots, m$.

Objective: steer x from x_0 (given) to x_1 (given) in minimal time.

Assume $\exists u$ that achieves the task in some time (for well-posedness.)

Bang-bang principle for linear systems cont.

Consider a system with general linear time-invariant dynamics

$$\dot{x} = Ax + Bu$$

where $x \in \mathbb{R}^m$ and $u_i \in [-1, 1], i = 1, \ldots, m$.

Hamiltonian: $H(x, u, p, p_0) = \langle p, Ax + Bu \rangle + p_0$

- Maximize $H \implies \langle p^*(t), b_i \rangle u_i^*(t) = \max_{|u_i| \leq 1} \langle p^*(t), b_i \rangle u_i(t)$.

$$u_i^*(t) = \operatorname{sgn}(\langle p^*(t), b_i \rangle) = \begin{cases} 1 & \text{if} \quad \langle p^*(t), b_i \rangle > 0, \\ -1 & \text{if} \quad \langle p^*(t), b_i \rangle < 0, \\ ? & \text{if} \quad \langle p^*(t), b_i \rangle = 0. \end{cases}$$

- The adjoint equation $\dot{p}^* = -A^T p^*$ allows us to investigate "?".

$\Rightarrow$ If (A, b_i) controllable (i.e., the system is **normal**), $\langle p^*(\cdot), b_i \rangle \not\equiv 0$ on any time interval. (finitely many switches)

Remark

If not all (A, b_i) controllable?

A bang-bang control is *a* time-optimal control for every linear control system and every U that is a convex polyhedron.

(can reach from x_0 to x_1 *in the same time* by other controls)

Minimum time-fuel problem

$$\text{minimize } J(u) = \int_{t_0}^{t_f} (1 + b|u(t)|)dt$$

$$\text{where } \ddot{x}_1 = u, \quad u \in [-1, 1].$$

- $x := (x_1, x_2)^T$, $\dot{x} = (x_2, u)^T$, $H = p_1 x_2 + p_2 u + p_0(1 + b|u|)$.
- The adjoint equation

$$\begin{pmatrix} \dot{p}_1^* \\ \dot{p}_2^* \end{pmatrix} = \begin{pmatrix} -H_{x_1}|_* \\ -H_{x_2}|_* \end{pmatrix} = \begin{pmatrix} 0 \\ -p_1^* \end{pmatrix} \Rightarrow \begin{pmatrix} p_1^* \\ p_2^* \end{pmatrix} = \begin{pmatrix} c_1 \\ -c_1 t + c_2 \end{pmatrix}.$$

- The Hamiltonian maximization.

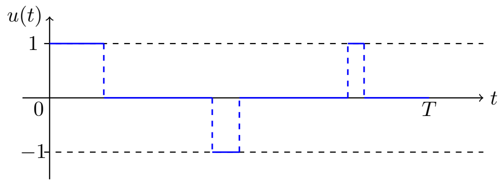
$$\Rightarrow u^*(t) = \begin{cases} -1 & \text{if } \; p_2^*(t) > b, \\ 0 & \text{if } \; -b < p_2^*(t) < b, \\ 1 & \text{if } \; p_2^*(t) < -b, \\ ? & \text{if } \; p_2^*(t) = \pm b. \end{cases}$$

Bang-off-bang control

Remarks

- $J(u) = \int_{t_0}^{t_f} (1 + b|u(t)|)dt$
- minimum time-fuel control, or finite horizon L^1-optimal control
- A **sparsity** property (directly related to L^0 "norm")
- L^1 optimality as a convex relaxation of L^0-optimal control problems.
- Maximum hands-off control (e.g., [Nagahara et al., TAC 16])
- Sparsity promoting control (e.g., [Jovanovic and Lin, ECC 13])

Maximum hands-off control (scalar)



Define $\|u\|_0 \triangleq m_L(\text{supp}(u))$ (the length of the support of u)

- Maximum hands-off control: $J_0(u) = \frac{1}{T}\|u\|_0$
- L^1-Optimal Control: $J_1(u) = \frac{1}{T}\int_0^T |u(t)|dt$
- The solution set of the maximum hands-off control problem is equivalent to that of L^1-optimal control problem under the normality assumption. [M. Nagahara et al. TAC, vol. 61, no. 3, 2016]

Fuller's problem

$$\text{minimize } J(u) = \int_{t_0}^{t_f} x_1^2(t)dt$$

where $(\dot{x}_1, \dot{x}_2)^T = (x_2, u)^T, \quad u \in [-1, 1], \quad S = [t_0, \infty) \times \left\{(0, 0)^T\right\}$.

$$H = p_1 x_2 + p_2 u + p_0 x_1^2.$$

We again have

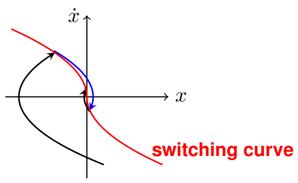
$$u^*(t) = \text{sgn}(p_2^*(t)) = \begin{cases} 1 & \text{if } \; p_2^*(t) > 0, \\ -1 & \text{if } \; p_2^*(t) < 0, \\ ? & \text{if } \; p_2^*(t) = 0, \end{cases}$$

but the adjoint equation is different to the time-optimal control:

$$\dot{p}_1^* = -2p_0^* x_1^*, \quad \dot{p}_2^* = -p_1^*.$$

Fuller's problem cont.

- Optimal controls are bang-bang with infinitely many switches.
- Switching takes place on the curve $\{(x_1, x_2)^T : x_1 + \gamma|x_2|x_2 = 0\}$ where $\gamma \approx 0.445$.
- Time intervals between consecutive switches decrease in geometric progression.
- Fuller's phenomenon, Zeno behaviour, or chattering.



- $J(u) = \int_{t_0}^{t_f} |x_1(t)|^\nu dt$
- $\nu \in [0, \bar{\nu}]$: at most one switch, $\nu > \bar{\nu}$: Zeno behaviour. $\bar{\nu} \approx 0.35$.

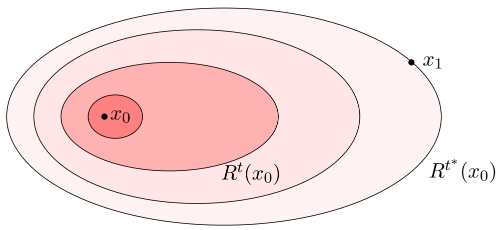
Existence of optimal control

- Perron's paradox: *Let N be the largest integer. If $N > 1$, then $N^2 > N$ contradicting the definition of N. Hence $N = 1$.*
- Does an optimal solution exist to our problem?
- At least one control must exist that drives (t_0, x_0) to S. – controllability.
- Is this enough?
- Example: minimal-time control for $\dot{x} = u \in \mathbb{R}, x_0 = 0, x_1 = 1$. $U = \mathbb{R}$ is **unbounded**.
- How about the case $u \in [0, 1)$? $U = [0, 1)$ is **not closed**.

Compact reachable sets

- $R^t(x_0)$: the set of points reachable from $x(t_0) = x_0$ at time $t \geq t_0$. (U given)
- $R^t(x_0)$ must be **compact**, i.e., bounded and closed.
- $t^* - t_0$ fastest transfer time $\Rightarrow x_1 \in \partial R^{t^*}(x_0)$

What are the conditions to guarantee the compactness of $R^t(x_0)$?



Filippov's theorem

Filippov's theorem

Given a control system $\dot{x} = f(t, x, u)$, $x(t_0) = x_0$ with $u \in U$, assume that

- its solutions exist on $[t_0, t_f]$ for all controls $u(\cdot)$ and
- for every pair (t, x) the set $\{f(t, x, u) : u \in U\}$ is compact and convex.

Then $R^t(x_0)$ is compact for each $t \in [t_0, t_f]$.

- A sufficient condition for compactness of reachable sets.
- applies to, e.g., $\dot{x} = f(x) + G(x)u$ with *compact and convex* U.
- For linear systems $\dot{x} = Ax + Bu$, $R^t(x_0)$ is compact if U *is compact and convex.*

Existence of time-optimal controls for linear systems

Consider the linear control system

$$\dot{x} = Ax + Bu$$

$u \in U$ *compact and convex.*

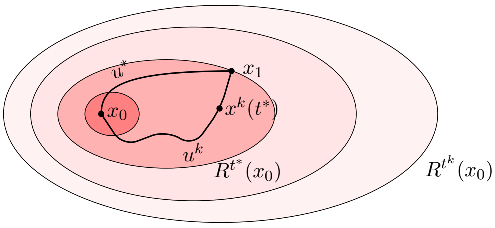
Objective: steer x from given $x(t_0) = x_0$ to given x_1 in minimal time.

$x_1 \in R^t(x_0)$ for some $t \geq t_0 \implies$ a time-optimal control exists.

Sketch of proof

Let $t^* = \inf\{t \geq t_0 : x_1 \in R^t(x_0)\}$. If $x_1 \in R^{t^*}(x_0)$, we are done.

- $\exists t_k \searrow t^*$ s.t. $x_1 \in R^{t_k}(x_0)$ with a corresponding u_k s.t. $x^k(t_k) = x_1$.
- Show that $x^k(t^*) \to x_1$ as $t_k \to t^*$.
- $\Rightarrow x_1 \in R^{t^*}(x_0)$ since the closedness property of $R^{t^*}(x_0)$ guaranteed by Filippov's theorem.



Information on Mini-project

- Date: March 20 (Tue)?
- Formulate your own optimal control problem.
- You can pair up.
- Solve the problem numerically. JModelica, or your preferred method.