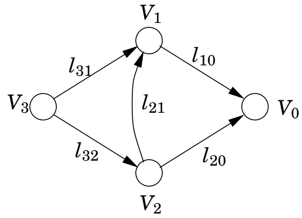
L4: Hybrid systems and dynamic programming

- **Hybrid Systems**
  - Piecewise Linear Systems
  - Piecewise Quadratic Lyapunov Functions
  - Value Iteration
  - Policy Iteration
  - Jump Linear Systems

Literature:

Piecewise Quadratic: Johansson/Rantzer, IEEE TAC, 43:4 (1998)
Networked Control Example: Nilsson/B/W, Automatica 34:1 (1998)
Value and policy iteration:
`web.mit.edu/dimitrib/www/Det_Opt_Control_Lewis_Vol.pdf`

Optimal transportation in a graph



Minimize $l_{31}\rho_{31} + l_{32}\rho_{32} + l_{21}\rho_{21} + l_{10}\rho_{10} + l_{20}\rho_{20}$

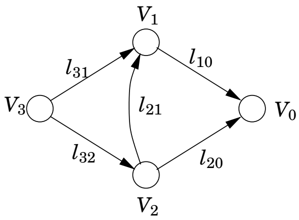
subject to $\rho_{31}, \ldots, \rho_{20} \geq 0$

$\rho_{31} + \rho_{32} \geq 1$

$-\rho_{31} - \rho_{21} + \rho_{10} \geq 1$

$-\rho_{32} + \rho_{21} + \rho_{20} \geq 1$

Dual gives lower bounds



Maximize $V_1 + V_2 + V_3$

subject to $V_3 - V_1 \leq l_{31}$ $\quad V_0 = 0$

$V_3 - V_2 \leq l_{32}$

$\vdots$

$V_2 - V_0 \leq l_{20}$

Continuous Optimal Control

Minimize $\int_{t_0}^{t_1} l(x,u)dt$

subject to $\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ x(t_0) = x_0, \quad x(t_1) = x_1 \end{cases}$

where $x(t) \in Q,\ u(t) \in \Omega.$

Lower Bounds on the Optimal Cost

Bellman's inequality:

Maximize $V(x_0) - V(x_1)$

subject to $0 \leq \frac{\partial V}{\partial x} f(x,u) + l(x,u) \qquad \forall x \in Q,\ u \in \Omega$

The constraint gives a lower bound on the achievable cost:

$$V(x_0) - V(x_1) = -\int_{t_0}^{t_1} \frac{\partial V}{\partial x} f(x,u)dt \leq \int_{t_0}^{t_1} l(x,u)dt$$

General Hybrid Systems

A system involving both discrete and continuous dynamics

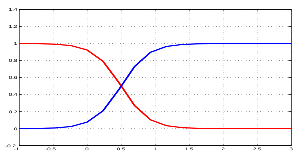
$$\begin{cases} \dot{x}(t) = f_{q(t)}(x(t), u(t)) \\ q(t) = \nu(q(t^-), \mu(t^-)) \end{cases} \qquad q \in Q = \{1, 2, \ldots, N\}$$

An optimization criterion takes the form

$$J(x_0, q_0) = \int_{t_0}^{t_f} l_q(x,u)dt + \sum_{k=1}^{M} s(x(t_k^-), q(t_k^-), q(t_k^+)),$$

Example — A car with two gears

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = g_q(x_2)u, \quad q = 1,2 \ \ |u| \leq 1 \end{cases}$$

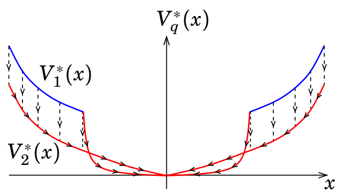


Bring the car to $x_f = (0,0)$, $q_f = 1$, while minimizing $J(x_0, q_0) = \int_{t_0}^{t_f} 1dt + \sum_{k=1}^{M} 1$

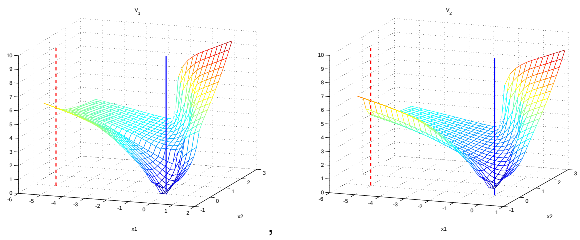
Bellmans Inequality for Hybrid Systems

$V_{q_0}(x_0)$ is a lower bound on the optimal cost for bringing the system from (x_0, q_0) to (x_f, q_f) if $V_{q_f}(x_f) = 0$ and

$$0 \leq \frac{\partial V_q(x)}{\partial x} f_q(x,u) + l_q(x,u)$$
$$0 \leq V_q(x) - V_r(x) + s(x, r, q)$$

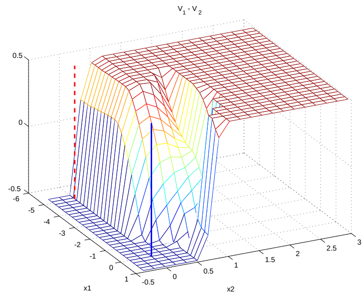


Optimal costs for the two initial gears



When should one switch gears?

Study the difference between V_1 and V_2:



L4: Hybrid Systems and piecewise linear systems

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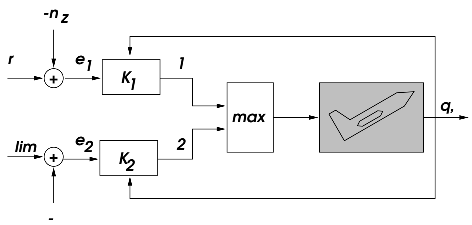
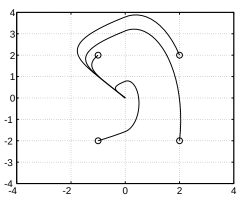
Aircraft Example



(Branicky, 1993)

Phase Plane

$$\dot{x} = \begin{cases} A_1 x \text{ for } x_1 \geq 0 \\ A_2 x \text{ for } x_1 \leq 0 \end{cases} \qquad A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix}$$



No common *quadratic* Lyapunov function exists.

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Piecewise quadratic Lyapunov Function for Aircraft

$$V(x) = \begin{cases} x^* P x & \text{if } x_1 < 0 \\ x^* P x + \eta x_1^2 & \text{if } x_1 \geq 0 \end{cases}$$

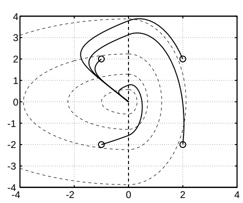
The matrix inequalities

$$\begin{aligned} A_1^* P + P A_1 &< 0 \\ P &> 0 \\ A_2^*(P + \eta E^* E) + (P + \eta E^* E) A_2 &< 0 \\ P + \eta E^* E &> 0 \end{aligned}$$

with $E = [1\ 0]$, have the solution $P = \text{diag}\{1, 3\}$, $\eta = 7$.

Phase Plot for Aircraft

A *piecewise quadratic* Lyapunov function exists!



General Piecewise Affine Systems

Space partition $\cup_{i\in I} X_i$ where $I = I_0 \cup I_1$ and $0 \notin X_i$ for $i \in I_1$.
With $a_i = c_i = 0$ for $i \in I_0$, let

$$\begin{cases} \dot{x} = a_i + A_i x + B_i u \\ y = c_i + C_i x + D_i u \end{cases} \qquad \text{for } x \in X_i \qquad (1)$$

$$\left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] = \left[\begin{array}{cc|c} A_i & a_i & B_i \\ 0 & 0 & 0 \\ \hline C_i & c_i & D_i \end{array}\right] \qquad \bar{x} = \begin{bmatrix} x \\ 1 \end{bmatrix}$$

Then

$$\left[\begin{array}{c} \dot{\bar{x}} \\ \hline y \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_i & \bar{B}_i \\ \hline \bar{C}_i & \bar{D}_i \end{array}\right] \left[\begin{array}{c} \bar{x} \\ \hline u \end{array}\right]$$

Piecewise Quadratic Functions

Let $\bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix}$ with $f_i = 0$ for $i \in I_0$ and

$$\bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} = \bar{F}_j \begin{bmatrix} x \\ 1 \end{bmatrix}, \quad x \in X_i \cap X_j \quad i, j \in I$$

Then, for every matrix T, the function

$$V(x) = \begin{bmatrix} x \\ 1 \end{bmatrix}^* \bar{F}_i' T \bar{F}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \text{for } x \in X_i$$

is piecewise quadratic and continuous.

S-procedure

Let $\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix}$ with $e_i = 0$ for $i \in I_0$ and

$$E_i x + e_i \geq \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \qquad x \in X_i$$

If there exist matrices W_i with non-negative entries such that

$$\epsilon I < \bar{P}_i + \bar{E}_i' W_i \bar{E}_i \qquad \forall i \in I$$

then

$$\epsilon |x|^2 < \begin{bmatrix} x \\ 1 \end{bmatrix}^* \bar{P}_i \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad \text{for } x \in X_i$$

Stability Theorem

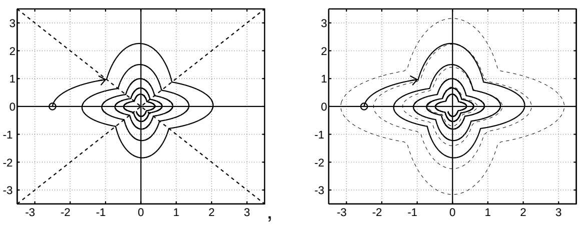
Consider symmetric matrices T, U_i and W_i, such that U_i and W_i have non-negative entries, while $P_i = F_i^T T F_i$ and $\bar{P}_i = \bar{F}_i^T T \bar{F}_i$ satisfy

$$\begin{cases} 0 > A_i' P_i + P_i A_i + E_i' U_i E_i \\ 0 < P_i - E_i' W_i E_i \end{cases} \qquad i \in I_0$$

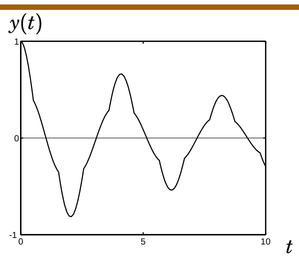
$$\begin{cases} 0 > \bar{A}_i' \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i' U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i' W_i \bar{E}_i \end{cases} \qquad i \in I_1$$

Then $x(t)$ tends to zero exponentially for every continuous piecewise C^1 trajectory in $\cup_{i\in I} X_i$ satisfying (1) with $u \equiv 0$.

Flower Example



Observability



$$\begin{cases} \dot{x}(t) = A_{i(t)} x(t) \\ y(t) = C_{i(t)} x(t) \end{cases} \qquad \text{for } x(t) \in X_{i(t)}$$

Estimate $\int_0^\infty |y|^2 dt$ given $x(0)$

Bounds on Transient Energy

Suppose $x(0) \in X_{i_0}$ and $x(\infty) = 0$. If U_i has non-negative entries, while $P_i = F_i' T F_i$ and $\bar{P}_i = \bar{F}_i' T \bar{F}_i$ satisfy

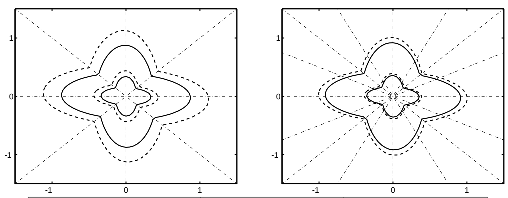
$$0 > P_i A_i + A_i' P_i + C_i' C_i + E_i' U_i E_i \qquad i \in I_0$$
$$0 > \bar{P}_i \bar{A}_i + \bar{A}_i' \bar{P}_i + \bar{C}_i' \bar{C}_i + \bar{E}_i' U_i \bar{E}_i \qquad i \in I_1$$

then

$$\int_0^\infty |y|^2 dt \leq \inf_{T, U_i} \begin{bmatrix} x(0) \\ 1 \end{bmatrix}' P_{i_0} \begin{bmatrix} x(0) \\ 1 \end{bmatrix}$$

The opposite inequalities give a lower bound.

Flower Example Revisited



Number of Cells	Lower bound	Upper bound
4	0.60	2.50
8	1.33	2.18
16	1.65	1.98
32	1.78	1.88

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Optimal Control

Idea:
Formulate a control synthesis problem in terms of optimization

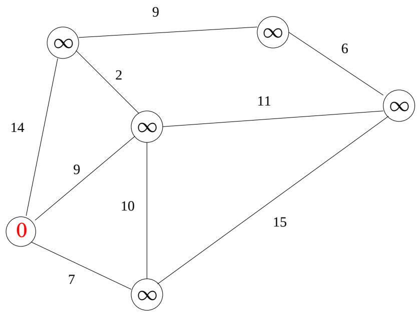
+ Gives systematic design procedure
+ Can be used on nonlinear models
+ Can capture limitations as constraints
– Hard to find suitable criterium?!
– Can be hard to find the optimal controller

Dynamic Programming, Richard E. Bellman 1957

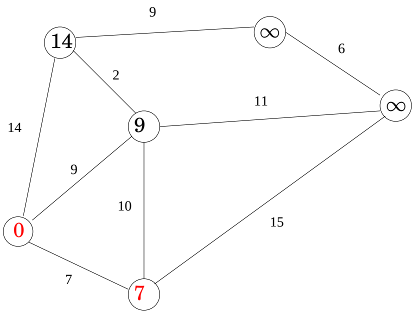
An optimal trajectory on the time interval $[T_1, T]$ must be optimal also on each of the subintervals $[T_1, T_1+\epsilon]$ and $[T_1+\epsilon, T]$.

Example: Dijkstra's Algorithm

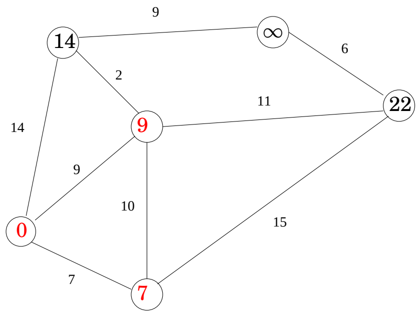
For each node, find the shortest path to the goal!



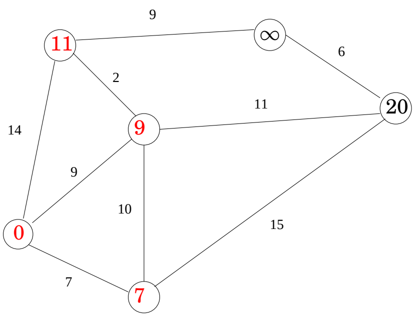
Example: Dijkstra's Algorithm



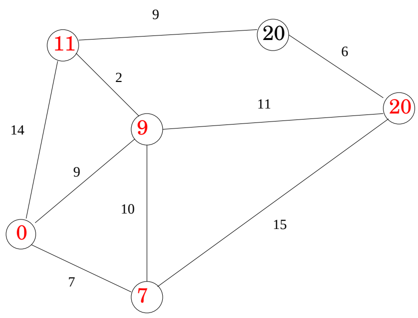
Example: Dijkstra's Algorithm



Example: Dijkstra's Algorithm



Example: Dijkstra's Algorithm



Dynamic Programming in Discrete Time

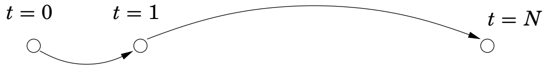
Minimize $\sum_{t=0}^{N-1} g(x(t), u(t))$

subject to $x(t+1) = f(x(t), u(t)) \quad t = 0, 1, 2, \dots, N-1$

$x(0) = x_0$

Let $J_N(x_0)$ denote the minimal value. The value function J_N satisfies the *Bellman equation*

$$J_N(x) = \min_u [g(x,u) + J_{N-1}(f(x,u))]$$



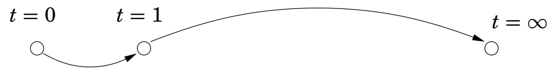
Infinite horizon Bellman equation

Minimize $\sum_{t=0}^{\infty} g(x(t), u(t))$

subject to $x(t+1) = f(x(t), u(t)) \quad x(0) = x_0$

Let $J^*(x_0)$ denote the minimal value. The value function J^* satisfies the *Bellman equation*

$$J^*(x) = \min_u [g(x,u) + J^*(f(x,u))]$$



Value iteration

Recall that

$$J_N(x) = \min_u [g(x,u) + J_{N-1}(f(x,u))]$$

Starting from $J_0(x) \equiv 0$ the iteration gives $J_N(x)$ with

$$0 \le J_1(x) \le J_2(x) \le \cdots \le J^*(x)$$
$$\lim_{N\to\infty} J_N(x) \to J^*(x), \quad N \to \infty$$

Useful for finite X. Often extremely complex when $X = \mathbf{R}^n$.

Theorem: Value Iteration Convergence

Suppose the condition $0 \le J^*(f(x,u)) \le \gamma g(x,u)$ holds uniformly for some $\gamma < \infty$.

Then $J_N \to J^*$ according to the inequalities

$$\left[1 - \frac{1}{(1+\gamma^{-1})^N}\right] J^*(x) \le J_N(x) \le J^*(x)$$

Proof idea

Use the assumption $J^*(f(x,u)) \le \gamma g(x,u)$ to get

$$\begin{aligned} J_1(x) &= \min_u g(x,u) \\ &= (1+\gamma)^{-1} \min_u [\gamma g(x,u) + g(x,u)] \\ &\ge (1+\gamma)^{-1} \min_u [J^*(f(x,u)) + g(x,u)] \\ &= (1+\gamma)^{-1} J^*(x) \end{aligned}$$

recursively gives the inequalities for $J_2, J_3, \dots$

Dynamic Programming in Continuous Time

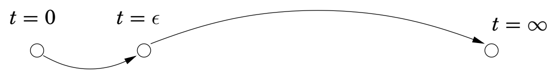
Minimize $\int_0^{\infty} g(x(t), u(t))dt$

subject to $\dot{x}(t) = f(x(t), u(t)) \quad t \ge 0$

$x(0) = x_0$

Let $J^*(x_0)$ denote the minimal value. The value function J^* satisfies the *Bellman equation*

$$0 = \min_u \left[g(x,u) + \frac{\partial J^*}{\partial x} f(x,u)\right]$$



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Policy Iteration

For a control policy $\mu : x \to u$, define

$$J_\mu(x_0) = \sum_{t=0}^{\infty} g(x(t), u(t))$$

where $x(t+1) = f(x(t), \mu(x(t))) \quad x(0) = x_0 \in X$

Assuming that $g \ge 0$, this gives

$$J_\mu(x) = g(x, \mu(x)) + J_\mu(f(x, \mu(x)))$$

Starting with any policy $\mu^0 : x \to u$, the *policy iteration* algorithm is defined by

$$\mu^{k+1} = \arg\min_u \left\{ g(x,u) + J_{\mu^k}(f(x,u)) \right\}$$

Policy Iteration Convergence

The policy iteration algorithm

$$\mu^{k+1} = \arg\min_u \left\{ g(x,u) + J_{\mu^k}(f(x,u)) \right\}$$

gives

$$J_{\mu^k}(x) \geq g(x, \mu^{k+1}(x)) + J_{\mu^k}(f(x, \mu^{k+1}(x)))$$

Hence $J_{\mu^k} \geq J_{\mu^{k+1}}$. Repeated use of this argument gives

$$J_{\mu^0}(x) \geq J_{\mu^1}(x) \geq J_{\mu^2}(x) \geq \ldots \geq J^*(x)$$

Under general conditions [Puterman/Brumelle 1979] proved that the convergence of policy is superlinear, i.e.

$$\limsup_{k\to\infty} \frac{J_{\mu^{k+1}}(x) - J^*(x)}{J_{\mu^k}(x) - J^*(x)} = 0$$

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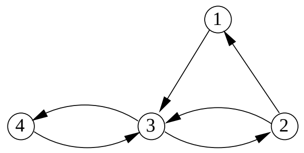
Jump Linear Systems



Let $\theta(0), \theta(1), \theta(2), \ldots \in \{1, \ldots, N\}$ be a Markov process. Given matrices $A_\theta, B_\theta, C_\theta$ for every $\theta \in \{1, \ldots, N\}$, the (stochastic) linear system

$$\begin{aligned} x(t+1) &= A_{\theta(t)} x(t) + B_{\theta(t)} u(t) \\ y(t) &= C_{\theta(t)} x(t) \end{aligned}$$

is called a *jump linear system*.

Jump Linear Systems

A jump linear system called *mean square stable* if $\lim_{t\to\infty} \mathbf{E}|x|^2 = 0$ when $u \equiv 0$.

Let q_{ij} be the transition probability from $\theta = j$ to $\theta = i$. Then mean square stability is equivalent to existence of (covariance matrices) $X_1, \ldots, X_N \succ 0$ such that

$$X_i \succ \sum_{ij} q_{ij} A_j X_j A_j^T.$$

Equivalently there exist (cost) matrices $P_1, \ldots, P_N \succ 0$ such that

$$P_i \succ \sum_{ij} q_{ij} A_j^T P_j A_j.$$

Optimal Control of Jump Linear Systems

Consider a jump linear system with transition probabilitis q_{ij}:

$$\begin{aligned} x(t+1) &= A_{\theta(t)} x(t) + B_{\theta(t)} u(t) \\ y(t) &= C_{\theta(t)} x(t) \end{aligned}$$

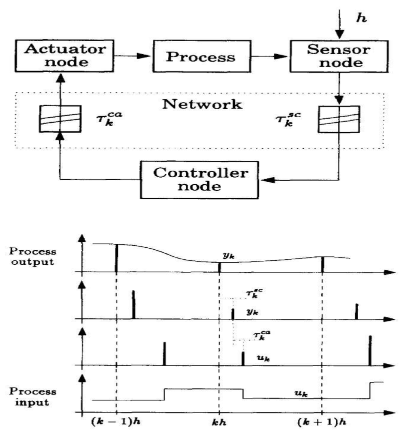
Problem:
Find a control law $u = -L_\theta x$ to minimize $\mathbf{E}(|x|^2 + |u|^2)$.

Solution:
the optimal control law is obtained from Bellman's equation:

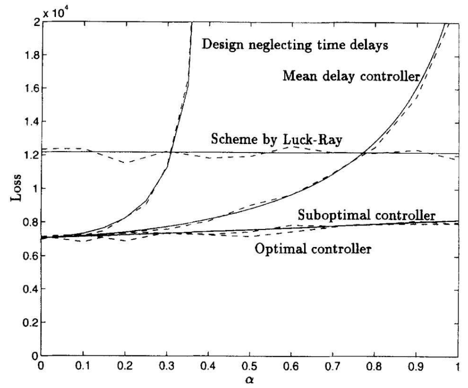
$$x^T P_\theta x = \min_u \left[|x|^2 + |u|^2 + \sum_i q_{i\theta}(A_i x + B_i u)^T P_i (A_i x + B_i u) \right]$$

$$u = \underbrace{\arg\min_u \left[|x|^2 + |u|^2 + \sum_i q_{i\theta}(A_i x + B_i u)^T P_i (A_i x + B_i u) \right]}_{-L_\theta x}$$

Example: Networked Control



Networked Control as a Jump Linear System



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