L3: Density functions and sum-of-squares methods

- Lyapunov Stabilization Computationally Untractable
- Density Functions
- “Almost” Stabilization Computationally Convex
- Duality Between Cost and Flow
- Sum-of-squares Optimization
- Examples

Literature.
Density functions: Rantzer, Systems & Control Letters, 42:3 (2001)
Synthesis: Prajna/Parrilo/Rantzer, TAC 49:2 (2004)
SOSTOOLS and its Control Applications, Prajna/P/S/P (2005)

Control and stabilization

Problem: Given functions $f(x)$ and $g(x)$ find $u(x)$ such that the differential equation

$$\dot{x} = f(x) + g(x)u(x)$$

has a globally asymptotically stable equilibrium in $x = 0$.

Unfortunately, the search for (V, u) such that

$$\frac{\partial V}{\partial x}[f + gu] < 0$$

is non-convex and difficult.

Non-connected set of Lyapunov functions

Every continuous stabilizing control law $u(x)$ for the system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}[u(x) - 3x_1](x_2)^2/|x|^2\\ u(x)\end{bmatrix}$$

must have the property that $u(x)$ has constant sign along the half line $x_1 > 0$, $x_2 = 0$. Zero crossing would create a second equilibrium. A Lyapunov function satisfies

$$0 > \nabla V \cdot f(x, u) = \frac{\partial V}{\partial x_2}u(x) \qquad \text{for } x_1 > 0,\ x_2 = 0$$

so also $\partial V/\partial x_2$ must have constant non-zero sign along the same half line.

$u_1(x) = -3x_1 - 6x_2$ is stabilizing with $V_1(x) = x_1^2 + x_2^2 + x_1x_2$.
$u_2(x) = x_1 - 2x_2$ is stabilizing with $V_2(x) = x_1^2 + x_2^2 - x_1x_2$.

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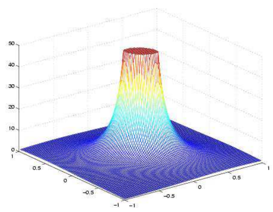
A criterion for almost global attractivity

Given $\dot{x}(t) = f(x(t))$, where $f \in \mathbf{C}^1(\mathbf{R}^n, \mathbf{R}^n)$ and $f(0) = 0$, suppose there exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ with $\rho(x)f(x)/|x|$ integrable on $\{x \in \mathbf{R}^n : |x| \geq 1\}$ and

$$[\nabla \cdot (\rho f)](x) > 0 \qquad \text{for almost all } x \neq 0$$

Then, for almost all initial states $x(0)$ the trajectory $x(t)$ tends to zero as $t \to \infty$.



Proof idea

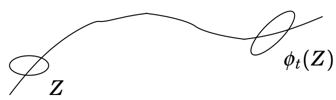
For $x_0 \in \mathbf{R}^n$, let $\phi_t(x_0)$ for $t \geq 0$ be the solution $x(t)$ of

$$\frac{dx}{dt} = f(x) \qquad\qquad x(0) = x_0$$

Liouville's theorem gives

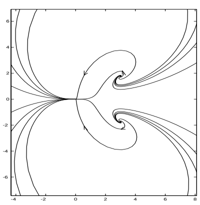
$$\int_{\phi_t(Z)} \rho(x)dx - \int_Z \rho(z)dz = \int_0^t \int_{\phi_\tau(Z)} [\nabla \cdot (\rho f)](x)dxd\tau$$

Every invariant set outside a neighborhood of zero must be of measure zero.



Example

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2x_1 + x_1^2 - x_2^2\\ -6x_2 + 2x_1x_2\end{bmatrix}$$



The system has four equilibria $(0, 0)$, $(2, 0)$ and $(3, \pm\sqrt{3})$. Let $\rho(x) = |x|^{-4}$. Then

$$[\nabla \cdot (\rho f)](x) = 16x_2^2|x|^{-6}$$

Exceptional Trajectories: The three unstable equilibria, the axis $x_2 = 0$, $x_1 \geq 2$ and the stable manifold of the equilibrium $(2, 0)$, that spirals out from the equilibria $(3, \pm\sqrt{3})$.

A Converse Theorem

Let $f \in \mathbf{C}^2(\mathbf{R}^n, \mathbf{R}^n)$ and $f(x)/|x|$ bounded and suppose that $x = 0$ is a stable equilibrium of the system $\dot{x} = f(x)$. Then, the following two conditions are equivalent.

- For almost all initial states $x(0)$ the solution $x(t)$ tends to zero as $t \to \infty$.
- There exists a non-negative $\rho \in \mathbf{C}^1(\mathbf{R}^n \setminus \{0\}, \mathbf{R})$ which is integrable outside a neighborhood of zero and such that

$$[\nabla \cdot (f\rho)](x) > 0 \qquad \text{for almost all } x$$

From Lyapunov function to density function

Let $V(x) > 0$ for $x \neq 0$ and

$$\nabla V \cdot f < \alpha^{-1}(\nabla \cdot f)V$$

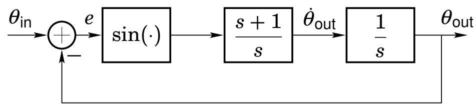
for some $\alpha > 0$. Then $\rho(x) = V(x)^{-\alpha}$ satisfies $[\nabla \cdot (\rho f)](x) > 0$.

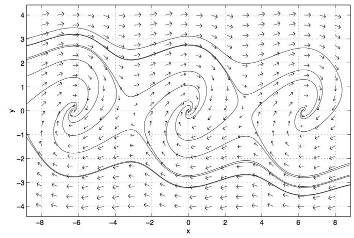
In particular, if P is a positive definite matrix satisfying

$$A^T P + PA < (\alpha^{-1} \operatorname{trace} A)P$$

then $\rho(x) = (x^T P x)^{-\alpha}$ can be used.

Example — PLL with two integrators





$$\frac{d^2 e}{dt^2} = -\sin e - \cos e \frac{de}{dt}$$

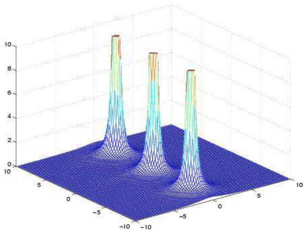
Tools for *almost global stability* needed!

Example — PLL with two integrators

$$\begin{bmatrix} \dot e \\ \dot y \end{bmatrix} = f(e, y) = \begin{bmatrix} y \\ -\sin e - y \cos e \end{bmatrix}$$

$$\rho(e, y) = \frac{1}{\psi} = \frac{1}{y^2 + 2 - 2\cos e + y \sin e} \geq 0$$

$$(\nabla \cdot f\rho) = \rho^2(1 - \cos e)^2 \geq 0$$



A warning example (by David Angeli)

Consider a pendulum with damping

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = f(x, y) = \begin{bmatrix} y \\ -\sin x - y \end{bmatrix}$$

A density function proving almost global attractivity must satisfy

$$0 \leq \nabla \cdot (f\rho) = \rho(\nabla \cdot f) + \dot\rho \qquad\qquad 0 \leq \rho$$

For the pendulum $\nabla \cdot f = -1 \leq 0$ everywhere.

Hence $\dot\rho \geq 0$ everywhere, with equality at unstable equilibria.

In particular, $\rho = 0$ on the stable manifold of the unstable (upright) equilibrium! This makes it virtually *impossible* to find ρ by numerical optimization.

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Convex Control Synthesis

The search for (V, u) such that

$$\frac{\partial V}{\partial x}[f + gu] < 0$$

is difficult

The search for (ρ, u) such that

$$\nabla \cdot [(f + gu)\rho] > 0$$

is convex in the pair $(\rho, u\rho)$

If $\nabla \cdot [(f + gu_k)\rho_k] > 0$ for $k = 1, 2$, then $\nabla \cdot [(f + gu)(\rho_1 + \rho_2)] > 0$ for $u = (\rho_1 u_1 + \rho_2 u_2)/(\rho_1 + \rho_2)$.

Example — Patching nonlinear controllers

$$\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix} = \begin{bmatrix} [u(x) - 3x_1](x_2)^2/|x|^2 \\ u(x) \end{bmatrix}$$

Each of the controllers $u_1(x) = -3x_1 - 6x_2$ and $u_2(x) = x_1 - 2x_2$ gives global stability. Corresponding density functions are

$$\rho_1(x) = (x_1^2 + x_2^2 + x_1 x_2)^{-\alpha_1} \qquad \rho_2(x) = (x_1^2 + x_2^2 - x_1 x_2)^{-\alpha_2}$$

with α_1 and α_2 are sufficiently large. For $\alpha_1 > \alpha_2$

$$u(x) = \frac{\rho_1(x)u_1(x) + \rho_2(x)u_2(x)}{\rho_1(x) + \rho_2(x)}$$

acts as $u_1(x)$ for small x and as $u_2(x)$ for large x.

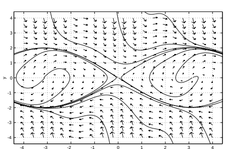
Example — Swing-up of inverted pendulum

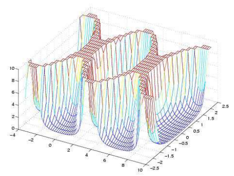
Dynamics and energy

$$\begin{bmatrix} \dot x \\ \dot y \end{bmatrix} = f_u(x, y) = \begin{bmatrix} y \\ \sin x + u \cos x \end{bmatrix} \qquad \begin{aligned} E &= y^2/2 + \cos x - 1 \\ \dot E &= uy\cos x \end{aligned}$$

The feedback $u_E = -y \cos x E$ steers towards the right energy.

$$\rho_0(x, y) = \frac{1}{E^2} \qquad \nabla \cdot (f_{u_E}\rho_0) = \frac{\cos^2 x}{E^2}\left(\frac{y^2}{2} + 1 - \cos x\right) \geq 0$$





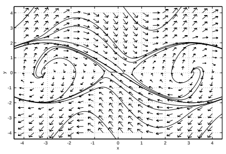
Example — Swing-up

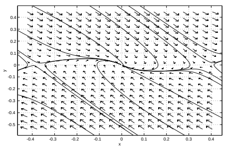
The controller $u_L(x,y) = -2\sin(x) - 2y$ is locally stabilizing with Lyapunov function
$V(x,y) = 8\sin(x/2)^2 + (2\sin(x/2)+y)^2$. With

$$\rho_E = \frac{1}{E^2 + \max(0, 1 - x^2 - y^2)} \qquad \rho_L = \max(0, V(x,y)^{-1} - 100)$$

$$u(x,y) = \frac{\rho_L}{\rho_E + \rho_L} u_L(x,y) + \frac{\rho_E}{\rho_E + \rho_L} u_E(x,y)$$

the pendulum swings up for almost all initial conditions.
Phase plots with smoothened max-operator:





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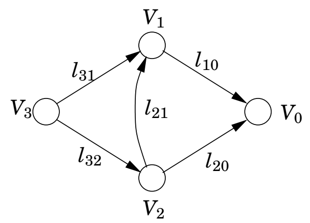
Linear Programming Duality

$$\begin{aligned} \min_x \quad & c^T x & = \quad \max_\lambda \quad & b^T \lambda \\ & b \le Ax & & c \ge A^T \lambda \\ & x \ge 0 & & \lambda \ge 0 \end{aligned}$$

Optimal transportation in a graph



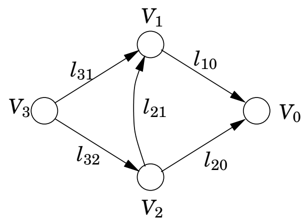
Minimize $\quad l_{31}\rho_{31} + l_{32}\rho_{32} + l_{21}\rho_{21} + l_{10}\rho_{10} + l_{20}\rho_{20}$
subject to $\quad \rho_{31}, \ldots, \rho_{20} \ge 0$

$\rho_{31} + \rho_{32} \ge 1$

$-\rho_{31} - \rho_{21} + \rho_{10} \ge 1$

$-\rho_{32} + \rho_{21} + \rho_{20} \ge 1$

Dual gives lower bounds



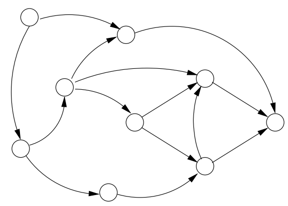
Maximize $\quad V_1 + V_2 + V_3$
subject to $\quad V_3 - V_1 \le l_{31} \qquad V_0 = 0$

$V_3 - V_2 \le l_{32}$

$\vdots$

$V_2 - V_0 \le l_{20}$

What do we learn from the graph problem?



- Two dual view-points
  - ρ gives an explicit control law
  - V gives a bound on the achievable cost
- Value iteration
- Decentralized computations
- Level set propagation

Linear Quadratic Gaussian Control

Consider the dynamics

$$x_{k+1} = Ax_k + Bu_k + w_k$$

Let w_k be white noise, independent of x_k, with covariance W.
Find a feedback law

$$u_k = -Lx_k$$

that minimizes the stationary covariance $\mathbf{E}\left\{ \begin{bmatrix} x_k \\ u_k \end{bmatrix}^T M \begin{bmatrix} x_k \\ u_k \end{bmatrix} \right\}$

Duality in linear quadratic control

The optimal cost has two dual expressions:

$$\max_P \text{trace}(PW) = \min_Q \text{trace}(QM)$$

where the maximization over $P \in \mathbf{R}^{n\times n}$ is such that

$$\begin{bmatrix} A^T \\ B^T \end{bmatrix} P \begin{bmatrix} A & B \end{bmatrix} - \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} + M \succeq 0$$

and the minimization over $Q \in \mathbf{R}^{(n+m)\times(n+m)}$ is such that

$$\begin{bmatrix} A & B \end{bmatrix} Q \begin{bmatrix} A^T \\ B^T \end{bmatrix} - \begin{bmatrix} I & 0 \end{bmatrix} Q \begin{bmatrix} I \\ 0 \end{bmatrix} + W \preceq 0$$

Notice: Q has an interpretation of the form $Q = \mathbf{E}\left\{ \begin{bmatrix} x \\ u \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}^T \right\}$

What do we learn from the linear case?

The duality is analogous to the graph problem:

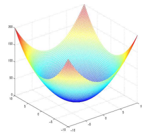
- Q gives an explicit control law $L = Q_{21}Q_{11}^{-1}$
- P gives a bound on the achievable cost

Duality in nonlinear control

For $\dot{x} = \sum_i u_i(x) f_i(x)$ let $V^*(x_0) = \inf_u \sum_i \int_0^\infty u_i(x) l_i(x) dt$. Then

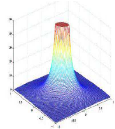
$$\sup_V \int_X \psi(x)V(x)dx = \int_X \psi(x)V^*(x)dx = \inf_{\rho_i} \sum_i \int_X l_i(x)\rho_i(x)dx$$

where sup is taken over non-negative V with



$$\nabla V \cdot f_i + l_i \geq 0$$
$$V(0) = 0$$

and inf is over ρ_i with $\rho_i > 0$ and



$$\sum_i \nabla \cdot (f_i(x)\rho_i(x)) \geq \psi(x)$$

(*Density functions* ρ_i correspond to control laws $u_i = \rho_i / \sum_i \rho_i$)

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Verify Positivty of a Polynomial

Does the polunomial

$$x^2y^2 + y^2 - 2xy - 4y + 5$$

take negative values?

No, because

$$x^2y^2 + y^2 - 2xy - 4y + 5 = (xy - 1)^2 + (y - 2)^2$$

How do we check if a polynomial can be written as a sum of squares?

Sum-of-squares Decomposition

To check if $2x^4 + 5y^4 - x^2y^2 + 2x^3y$ can be written as a sum of squares, note that

$$\begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}' \underbrace{\begin{bmatrix} q_{11} & q_{12} & q_{13} \\ q_{12} & q_{22} & q_{23} \\ q_{13} & q_{23} & q_{33} \end{bmatrix}}_{Q} \begin{bmatrix} x^2 \\ y^2 \\ xy \end{bmatrix}$$
$$= q_{11}x^4 + q_{22}y^4 + (q_{33} + 2q_{12})x^2y^2 + 2q_{13}x^3y + 2q_{23}xy^3$$

Use convex optimization to find $Q \succ 0$ subject ot the constraints that $q_{11} = 2$, $q_{22} = 5$, $q_{33} + 2q_{12} = -1$, $2q_{13} = 2$, $2q_{23} = 0$. Factorizing $Q = L^T L$ with

$$L = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

gives $2x^4 + 5y^4 - x^2y^2 + 2x^3y = \frac{1}{2}(2x^2 - 3y^2 + xy)^2 + \frac{1}{2}(y^2 + 3xy)$.

From LMIs to SOS

Linear Matrix Inequalities (LMIs): Optimization with constraints that certain *quadratic forms* must be *non-negative*:

Minimize $c_1u_1 + \cdots + c_nu_n$
subject to $A_0 + u_1A_1 + \cdots + u_nA_n \succeq 0$

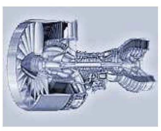
Sum-of-squares (SOS): Optimization with constraints that certain *polynomials* must be *sums of squares*:

Minimize $c_1u_1 + \cdots + c_nu_n$
subject to $A_0 + u_1A_1(x) + \cdots + u_nA_n(x)$ is a sum of squares.

Lyapunov

For $\dot{x} = f(x)$, a Lyapunov function must satisfy $V(x) \geq 0$, $\left(\frac{\partial V}{\partial x}\right)^T f(x) \leq 0$. Inequalities are *linear* in V.

A jet engine model (derived from Moore-Greitzer), with controller:



$$\dot{x} = -y + \frac{3}{2}x^2 - \frac{1}{2}x^3$$
$$\dot{y} = 3x - y;$$

A generic 4th order polynomial Lyapunov function.

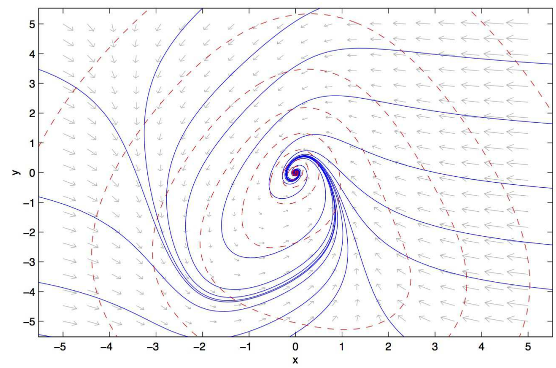
$$V(x, y) = \sum_{0 \leq j+k \leq 4} c_{jk} x^j y^k$$

Find a $V(x, y)$ by solving the SOS program:

$$V(x, y) \text{ is SOS}, \qquad -\nabla V(x, y) \cdot f(x, y) \text{ is SOS}.$$

Lyapunov example (cont.)

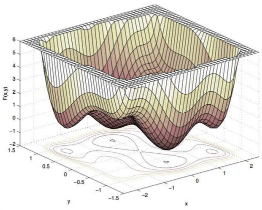
After solving, we obtain a Lyapunov function.



Global optimization

Consider $\min_{x,y} F(x,y)$, with
$F(x,y) := 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4$.

Not convex. Many local minima. NP-hard. How to find good lower bounds?



- Find the largest γ s.t.

$$F(x,y) - \gamma \text{ is SOS.}$$

- If exact, can recover optimal solution.
- Surprisingly effective.

Solving, the maximum γ is -1.0316. Exact bound.
Details in (P. & Sturmfels, 2001).

Direct extensions to constrained case.

Example: Numerical control synthesis

$$\rho(x) = \frac{a(x)}{b(x)^\alpha} \qquad u(x)\rho(x) = \frac{c(x)}{b(x)^\alpha}$$

$$\nabla \cdot [\rho(f+gu)] = \frac{1}{b^{\alpha+1}}[b\nabla \cdot (fa+gc) - \alpha \nabla b \cdot (af+gc)]$$

Select $b(x)$ as a quadratic Lyapunov function for a locally stabilizing controller.

Numerically computed control law

$$\begin{cases} \dot{x} = y - x^3 + x^2 + u \\ \dot{y} = x + 4u \end{cases}$$

Based on local analysis near $(x,y) = (0,0)$, we choose

$$b(x,y) = 3x^2 + 2xy + 2y^2$$

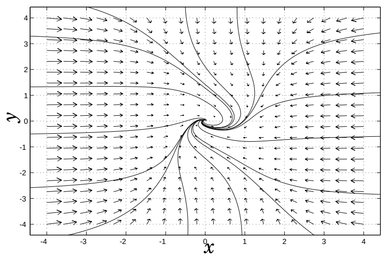
Let $a(x,y)$ be a constant and put $\alpha = 4$ to satisfy the integrability condition on $f\rho$.

Solving the inequality for $c(x,y)$ using SOSTOOLS gives

$$u(x,y) = \frac{c(x,y)}{a(x,y)} = -0.38x - 0.16y - 0.043y^3$$

The stabilized system

$$\begin{cases} \dot{x} = y - x^3 + x^2 + u \\ \dot{y} = x + 4u \\ u = -0.38x - 0.16y - 0.043y^3 \end{cases}$$



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Example: Attitude observer for a rigid body

The estimated attitude relative to the true attitude of a rigid body can be described by a matrix $R(t)$ which is orthogonal: $R(t)^T R(t) = I$. The estimate is correct when $R(t) = I$.

Consider an observer with error dynamics

$$\dot{R}(t) = kR(t)[R(t)^T - R(t)] + R(t)E(t)$$

where $E(t)$ represents measurement noise. The condition $E(t) = -E(t)^T$ guarantees that $R(t)$ stays orthogonal.

A Lyapunov Argument for Exact Measurements

For $E = 0$, the Lyapunov function $V(R) = \frac{1}{2}\|R - I\|^2$ satisfies $V \in [0,4]$ and

$$\frac{d}{dt}V(R(t)) = -\frac{k}{2}\|R(t) - R(t)^T\|^2$$

An orthogonal 3×3 matrix $R(t) \neq I$ can be symmetric only if it has two eigenvalues at -1, that is when $V(R)$ takes its maximal value 4.

Hence the Lyapunov function is strictly decreasing once $V < 4$. This proves almost global stability of the equilibrium $R = I$.

Density functions for observer dynamics

Analyzing $\dot{R} = f(R)$ with the density function ρ where

$$f(R) = kR(R^T - R) \qquad \rho(R) = \frac{1}{\|I - R\|^4}$$

gives

$$\nabla \cdot (\rho f) = \frac{2k}{\|I - R\|^4} > 0$$

so

$$\lim_{t\to\infty} R(t) = I$$

for almost all initial states $R(0)$.

Notice: Strict inequality gives robustness to measurement noise

Theorem on Rigid Body Observer

If $\|E(t)\| \leq \epsilon < \sqrt{6}k$ then almost all solutions to

$$\dot{R}(t) = kR(t)[R(t)^T - R(t)] + R(t)E(t)$$

converge towards a ball around the identity matrix I with radius

$$\sqrt{4 - 4\sqrt{1 - \epsilon^2/(8k^2)}}$$

[Vasconcelos/Rantzer/Silvestre/Oliveira, IEEE TAC, 56:11 (2011)]

L3: Density functions and computational methods

- Lyapunov Stabilization Computationally Untractable
- Density Functions
- “Almost” Stabilization Computationally Convex
- Duality Between Cost and Flow
- Sum-of-squares Optimization
- Example: Attitude observer for rigid body