Session 6

Nonlinear Controllability

Reading assignment

- Glad, Nonlinear Control Theory, Ch. 8 + Hörmander handout

Exercises marked with a "*" are more difficult

Exercise 6.1 Consider a car with N trailers. The front-wheels of the car can be controlled, and the car can drive forwards and backwards. Describe a manifold that can be used as state-space. Show that its dimension is $N+4$.

Exercise 6.2 Show that the sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ has the structure of a manifold with atlas, for example, consisting of two charts $(U_i, \phi_i, i = 1, 2)$ in stereographic projection.

Exercise 6.3 Use the implicit function theorem to show that under a certain condition (determine which) the root-locus, i.e. $\{s : a(s) + kb(s) = 0\}$, locally is a function of k. Determine a differential equation the branches $s_i(k)$ of the root locus satisfy.

Exercise 6.4 * Let $f_1(x), \ldots, f_n(x)$ be n vector fields that are smooth and linearly independent around $x_0 \in R^n$. Let $\Phi^t_{f_i}(x)$ be the corresponding transformation groups (defined at least for small t). Show that the transformations

$$
\begin{aligned}
X(t_1, \ldots, t_n) &= \Phi^{t_n}_{f_n} \circ \ldots \circ \Phi^{t_1}_{f_1}(x_0) \\
\widetilde{X}(t_1, \ldots, t_n) &= \Phi^1_{t_n f_n + \ldots + t_1 f_1}(x_0)
\end{aligned}
$$

both define smooth bijections between a neighborhood U of 0 and a neighborhood of x_0 (i.e. they are diffeormorphisms). These are nice ways to change coordinates (from x_i:s to t_i:s).

Exercise 6.5 Transform $X_p = x_2 \frac{\partial}{\partial x_1}$ to polar coordinates (r, φ). Calculate $X_p(f)$ when $p = (x_1, x_2) = (1, 1)$ and $f(x_1, x_2) = x_1^2 + x_2^2$.

Exercise 6.6 Let $X(x) = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}$. Write the vector field as a set of differential equations. Find $X^t(x_1, x_2)$. Calculate $X^t_* \frac{\partial}{\partial x_1}|_{(1,0)}$.

Exercise 6.7 The SISO system $\dot{x} = f(x) + g(x)u; \quad y = h(x)$ is said to have relative degree r at x_0 if

$$
\begin{aligned}
L_g L_f^i h(x) &\equiv 0 \quad \text{in a neighborhood of } x_0 \quad i = 0, \ldots, r-2 \\
L_g L_f^{r-1} h(x_0) &\neq 0
\end{aligned}
$$

Calculate the relative degree for

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} x_2 \\ -x_2 - x_1^3 - x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\
y &= x_1 \qquad \text{or} \quad y = x_2
\end{aligned}
$$

* Show also that this definition is consistent with the usual definition of relative degree for linear systems (as being the excess of poles over zeros). Can relative degree be changed by feedback $u = \alpha(x) + \beta(x)v$?

Exercise 6.8 * Let S^{2n-1} be the submanifold of R^{2n} generated by

$$x_1^2 + x_2^2 + \ldots + x_{2n}^2 = 1$$

Verify that the following defines a nowhere vanishing smooth vector field on S^{2n-1}:

$$\sum_{i=1}^{n} x_{2i} \frac{\partial}{\partial x_{2i-1}} - x_{2i-1} \frac{\partial}{\partial x_{2i}}$$

("It is possible to comb the hair on the S^{2n-1}-spheres.")

Exercise 6.9 Consider the vector fields

$$X = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}, \qquad Y = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}$$

Show that $[X, Y] = 0$. Interpret this geometrically.

Exercise 6.10 Using notation from the parking problem, calculate

[Glide, Steer]
[Glide, Drive]
[Glide, Wriggle]
[Rotate, Steer]
[Rotate, Drive]
[Rotate, Wriggle]
[Rotate, Glide]

Exercise 6.11 Write $[[f, g], [f, [f, g]]]$ as a linear combination of two brackets in the Lie-bracket tree.

Exercise 6.12 Show that the rolling penny system

$$\frac{d}{dt} \begin{pmatrix} x \\ y \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 1 \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2$$

has the accessibility property.

Exercise 6.13 Find a control that takes the system

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{x}_2 &= u_2 \\ \dot{x}_3 &= x_1 u_2 - x_2 u_1 \end{aligned}$$

from $(0, 0, 0)$ to $(0, 0, 1)$. Hint: Well chosen sinusoidals work.