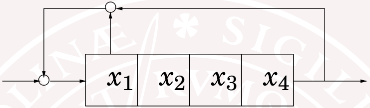
Lecture 4

- Realization from Weighting Pattern
- Minimal Realizations
- Realization from Transfer Function
- Realization from Markov Parameters
- Discrete Time

Rugh Ch 10, 11 (only pp194-199, skip proof of 11.7), (26)

Example: Shift Register Synthesis



$$
\begin{aligned}
x &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \\
x(k+1) &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k) \\
y(k) &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(k)
\end{aligned}
$$

Given a sequence $y(0), y(1), \ldots, y(N)$, what is the shortest shift register that can generate this output for the input $u \equiv 0$?

Definition: Realization

The state equation of dimension n

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \quad x(t_0) = 0 \\ y(t) &= C(t)x(t) \end{aligned}$$

is called a *realization* of the continuous *weighting pattern* $G(t,\sigma)$ if

$$G(t,\sigma) = C(t)\Phi(t,\sigma)B(\sigma) \qquad \forall t,\sigma$$

It is called *minimal* if no realization of smaller dimension exists.

Notice the distinction between the weighting pattern and the impulse response. The latter is zero for $t < \sigma$.

Theorem 1: Realizability

The weighting pattern $G(t,\sigma)$ has a realization of dimension n if and only if there exist matrix functions $H(t) \in \mathbf{R}^{p\times n}$, $F(t) \in \mathbf{R}^{n\times m}$ such that

$$G(t,\sigma) = H(t)F(\sigma) \quad \forall t,\sigma$$

Proof

If $G(t,\sigma) = H(t)F(\sigma)$, then

$$\begin{aligned} \dot{x}(t) &= F(t)u(t) \\ y(t) &= H(t)x(t) \end{aligned}$$

is a realization.

Conversely, if

$$G(t,\sigma) = C(t)\Phi(t,\sigma)B(\sigma),$$

then $G(t,\sigma) = H(t)F(\sigma)$ for

$$\begin{aligned} H(t) &= C(t)\Phi(t,0) \\ F(\sigma) &= \Phi(0,\sigma)B(\sigma) \end{aligned}$$

This does not work in discrete time. Why?

Warning

The realizations $\{0, F(t), H(t)\}$ are seldom "nice".

Consider $G(t,\sigma) = e^{-(t-\sigma)}$ with

$$\begin{cases} \dot{x}(t) & = e^{t} u(t) \quad \text{(unstable)} \\ y(t) & = e^{-t} x(t) \end{cases}$$

and compare with

$$\begin{cases} \dot{x}(t) & = -x(t) + u(t) \quad \text{(stable)} \\ y(t) & = x(t) \end{cases}$$

Theorem 2: Minimality

A linear realization of $G(t, \sigma)$ is minimal if and only if for some $t_0 < t_f$, it is both controllable and observable on (t_0, t_f).

Proof Omitted (see Rugh pp 162–164 if interested)

Remark

There may still exist realizations of the impulse-responses, i.e. for $t \geq \sigma$, of lower dimension. See Exercise 10.7.

Theorem 3: Periodic Realization

A periodic linear realization of $G(t, \sigma)$ exists if and only if it is realizable and $\exists T > 0$:

$$G(t+T, \sigma+T) = G(t, \sigma) \quad \forall t, \sigma$$

If so, then there also exists a minimal realization that is periodic.

The proof is omitted.

Theorem 4: LTI Realization

A linear time-invariant realization of $G(t,\sigma)$ exists if and only if G is realizable, continuously differentiable and

$$G(t,\sigma) = G(t-\sigma, 0)$$

Proof of Theorem 4

“Only if” is immediate. To prove “if” let $\{0, B(t), C(t)\}$ be a minimal realization. We want to find an LTI realisation. Introduce

$$A = -\int_{t_0}^{t_f} B'(\sigma)B(\sigma)^T d\sigma W(t_0,t_f)^{-1}$$

With $C(t)B(\sigma) = G(t-\sigma,0)$ it follows that

$$0 = \left[\frac{\partial}{\partial t}G(t-\sigma,0) + \frac{\partial}{\partial \sigma}G(t-\sigma,0)\right] B(\sigma)^T$$

$$= C'(t)B(\sigma)B(\sigma)^T + C(t)B'(\sigma)B(\sigma)^T$$

$$0 = \int_{t_0}^{t_f} \left[C'(t)B(\sigma)B(\sigma)^T + C(t)B'(\sigma)B(\sigma)^T\right] d\sigma$$

$$0 = C'(t) + C(t)\int_{t_0}^{t_f} B'(\sigma)B(\sigma)^T d\sigma W(t_0,t_f)^{-1}$$

$$0 = C'(t) - C(t)A, \quad C(t) = C(0)e^{At}$$

Proof of Theorem 4, cont'd

$$G(t,\sigma) = C(t)B(\sigma) = C(t-\sigma)B(0)$$
$$= C(0)e^{A(t-\sigma)}B(0)$$

A time-invariant realization is therefore

$$\dot{x} = Ax + B(0)u, \quad y = C(0)x$$

Example

The weighting pattern

$$G(t,\sigma) = e^{-(t-\sigma)^2}$$

satisfies $G(t,\sigma) = G(t-\sigma,0)$, but one can prove it is not factorizable as $F(t)H(\sigma)$, so no realization exists. In fact we have:

Remark

The weighting pattern $G(t,\sigma)$ is realizable as a time-invariant (finite-dimensional) system if and only if it can be written as

$$G(t,\sigma) = \sum_{k=1}^{n} \sum_{j=0}^{d_k-1} g_{kj} \cdot (t-\sigma)^j e^{\lambda_k(t-\sigma)}$$

Exercise

Write the time invariant impulse response

$$G(t,\sigma) = (t-\sigma)e^{-(t-\sigma)}$$

as

$$G(t,\sigma) = H(t)F(\sigma)$$

Th.5 Transfer Function Realizability

A transfer matrix $G(s)$ admits a linear time-invariant realization

$$G(s) = C(sI - A)^{-1}B$$

if and only if each entry of $G(s)$ is a strictly proper rational function.

Proof of Theorem 5

"Only if" is immediate.

To prove "if", choose $d(s) = s^r + d_{r-1}s^{r-1} + \cdots + d_0$ and write

$$d(s)G(s) = N_{r-1}s^{r-1} + \cdots + N_0$$

Let

$$\begin{aligned}
A &= \begin{bmatrix} 0 & I_m & 0 \\ 0 & & I_m \\ -d_0 I_m & -d_1 I_m & -d_{r-1} I_m \end{bmatrix} \\
B &= \begin{bmatrix} 0 & 0 & 0 & I_m \end{bmatrix}^T \\
C &= \begin{bmatrix} N_0 & N_1 & \dots & N_{r-1} \end{bmatrix} \\
Z(s) &= (sI - A)^{-1}B
\end{aligned}$$

Proof of Theorem 5

It is then easy to verify that

$$Z(s) \quad = \quad \frac{1}{d(s)} \begin{bmatrix} I_m \\ sI_m \\ \vdots \\ s^{r-1} I_m \end{bmatrix}$$

The equality $C(sI - A)^{-1}B = G(s)$ follows by left multiplication with C. Note: This realisation might not be minimal.

When $G(s)$ has distinct poles there is a more natural realization on diagonal form (which is minimal):

Gilbert-Realization

Introduce the partial fraction expansion

$$G(s) = \sum_{i=1}^{r} G_i \frac{1}{s - \lambda_i}$$

and the rank-factorizations

$$G_i = C_i B_i, \quad C_i \text{ is } p \times \rho_i, \quad B_i \text{ is } \rho_i \times m$$

where $\operatorname{rank} G_i = \rho_i$. Now use

$$\begin{aligned} A &= \text{diag}\{\lambda_1 I_{\rho_1}, \ldots, \lambda_r I_{\rho_r}\} \\ B &= \begin{bmatrix} B_1^T & \ldots & B_r^T \end{bmatrix}^T \\ C &= \begin{bmatrix} C_1, \ldots, C_r \end{bmatrix} \end{aligned}$$

That the realisation is minimal follows from the PBH-test.

Example

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{-1}{(s+1)(s+2)} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s+1}\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

with

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Theorem 6

$\{A, B, C\}$ is a minimal realisation of $G(s)$ if and only if it is controllable and observable.

Proof of Theorem 6

If $\{A, B, C\}$ is not a minimal realisation then there exists $\{F, G, H\}$ of dimension $n_z < n$ such that

$$g(t) = Ce^{At}B = He^{Ft}G \quad \forall t$$

This gives $CA^kB = g^{(k)}(0) = HF^kG \quad \forall k$, i.e.

$$\underbrace{\begin{bmatrix} C \\ \vdots \\ CA^{n-1} \end{bmatrix}}_{O_a} \underbrace{\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}}_{C_a} = \underbrace{\begin{bmatrix} H \\ \vdots \\ HF^{n-1} \end{bmatrix}}_{O_f} \underbrace{\begin{bmatrix} G & \cdots & F^{n-1}G \end{bmatrix}}_{C_f}$$

But O_f and C_f have rank less than or equal to n_z, so that holds also for either O_a or C_a. Therefore $\{A, B, C\}$ cannot be both controllable and observable.

Proof of Theorem 6, cont'd

Conversively, if $\{A, B, C\}$ is not controllable (similar if not observable) it can be transformed to

$$\left\{ \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \begin{bmatrix} C_1 & C_2 \end{bmatrix} \right\}$$

$$Ce^{At}B = C_1 e^{A_{11}t} B_1$$

so $\{A_{11}, B_1, C_1\}$ is a realization of lower dimension.

Theorem 7

Two minimal time-invariant realizations of $G(s)$ are related by a coordinate transformation $z = Px$.

The transformation is unique.

Proof of Theorem 7

Let the two minimal realizations be

$$g(t) = Ce^{At}B = He^{Ft}G \quad \forall t$$

With the notation from the proof of Theorem 6 let
$P = C_a C_f^T (C_f C_f^T)^{-1}$.

First prove that $P^{-1} = (O_f^T O_f)^{-1} O_f^T O_a$. The existence of the inverses are guaranteed by controllability and observability.

Then verify that $P^{-1}B = G$, $CP = H$ and $P^{-1}AP = F$.

For any other such transformation $\hat{P}$ it follows from
$O_a \hat{P} = O_f = O_a P$ and observability that $\hat{P} = P$.

Definition: Markov Parameters

Given a time-invariant impulse response $g(t)$, the corresponding *Markov parameters* are defined as

$$g(0), g'(0), g^{(2)}(0), g^{(3)}(0), \dots$$

Define also the block Hankel matrices (for $i, j \geq 0$)

$$\Gamma_{ij} = \begin{bmatrix} g(0) & g'(0) & \dots & g^{(j-1)}(0) \\ g'(0) & & & \\ \vdots & & \ddots & \\ g^{(i-1)} & & & g^{(i+j-2)}(0) \end{bmatrix}$$

We have $g^k(0) = CA^kB$ and

$$G(s) = g(0)s^{-1} + g'(0)s^{-2} + g^{(2)}(0)s^{-3} + \dots$$

Th. 8 Realization from Markov Parameters

An analytic impulse response $g(t)$ admits an n-th order time-invariant realization $\dot{x} = Ax + Bu, y = Cx$ if and only if there exist positive integers $l, k \leq n$ such that

$$\text{rank } \Gamma_{lk} = \text{rank } \Gamma_{l+1,k+j} = n, \quad j = 1, 2, \ldots$$

Proof Utilize

$$\Gamma_{ij} = M_i W_j$$

$$M_i = \begin{bmatrix} C \\ \vdots \\ CA^{i-1} \end{bmatrix}$$

$$W_j = \begin{bmatrix} B & AB & \cdots & A^{j-1}B \end{bmatrix}$$

like in the proof of Theorem 6. See Rugh 11.7 for details.

Example

What is the dimension of a minimial realisation of $g(t) = te^t$?

Since $g^{(k)}(0) = k$ we get

$$\operatorname{rank}\Gamma_{11} = \operatorname{rank}\begin{bmatrix}0\end{bmatrix} = 0$$

$$\operatorname{rank}\Gamma_{22} = \operatorname{rank}\begin{bmatrix}0 & 1\\ 1 & 2\end{bmatrix} = 2$$

$$\operatorname{rank}\Gamma_{3,k} = \operatorname{rank}\begin{bmatrix}0 & 1 & 2 & \dots\\ 1 & 2 & 3 & \dots\\ 2 & 3 & 4 & \dots\end{bmatrix} = \operatorname{rank}\begin{bmatrix}0 & 1\\ 1 & 1\\ 2 & 1\end{bmatrix} = 2, \quad k \geq 3$$

so the minimial dimension is 2. In fact, one can take

$$A = \begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad C = \begin{bmatrix}1 & 0\end{bmatrix}$$

Theorem 9 - Discrete Time

$$y(k) = \sum_{j=k_0}^{k} G(k,j)u(j)$$

$$G(k,j) = C(k)\Phi(k,j+1)B(j),\; k \geq j+1$$

Cannot define weighting pattern, that is $G(k,j)$ also for $k < j$, since Φ need not be invertible.

$$\exists H(k), F(k): \quad G(k,j) = H(k)F(j),\; k \geq j+1$$

$$\implies \quad \exists \text{ realization } \{A(k), B(k), C(k)\}$$

Proof

$$A(k) = I \Rightarrow \Phi(k,j+1) = I$$

Example

$$x(k+1) = u(k), \qquad y(k) = x(k)$$

is a realisation of

$$G(k,j) = \delta(k-j-1), \quad k \geq j+1$$

but you can not find a factorisation of the form

$$G(k,j) = H(k)F(j), \quad k \geq j+1$$

Example

$$x(k+1) = x(k) + \begin{bmatrix} 1 \\ \delta(k-1) \end{bmatrix} u(k)$$
$$y(k) = \begin{bmatrix} 1 & \delta(k) \end{bmatrix} x(k)$$

is reachable and observable on any interval containing $k = 0, 1, 2$, but it is not a minimal realisation of the pulse response

$$G(k,j) = 1 + \delta(k)\delta(j-1) = 1, \quad k \geq j+1$$

since

$$z(k+1) = z(k) + u(k), \qquad y(k) = z(k)$$

is of lower dimension.

Some things we (and Rugh) left out

We did not obtain a method to find a minimal (A, B, C, D) from a given $G(s)$ in the case of non-distinct poles. One solution is to use the non-minimal realisation in Theorem 5 and then apply Kalman decomposition (or balanced realisation). But there if of course a more direct approach see [Kailath, Linear Systems].

We could have talked about identification by state-space methods. See the course in Identification if interested.