Lecture 3

- Controllability
- Observability
- Controller and Observer Forms
- Balanced Realizations

Rugh, chapters 9, 13, 14 (only pp 247-249) and (25)
Controllability

How should **controllability** be defined?

Some (not used) alternatives:

By proper choice of control signal $u$

- any state $x_0$ can be made an equilibrium
- any state trajectory $x(t)$ can be obtained
- any output trajectory $y(t)$ can be obtained

The most fruitful definition has instead turned out to be the following
Controllability

The state equation

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \]

is called *controllable on* \((t_0, t_f)\), if for any \(x_0\), there exists \(u(t)\) such that \(x(t_f) = 0\) (“Controllable to origin”)

Question: Is this equivalent to the following definition:

“for \(x_0 = 0\) and any \(x_1\), there exists \(u(t)\) such that \(x(t_f) = x_1\)”

(“Controllable from origin”)

The audience is thinking!

Hint: \[ x(t_f) = \Phi(t_f, t_0)x(t_0) + \int_{t_0}^{t_f} \Phi(t_f, t)B(t)u(t)\,dt \]
The matrix function

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, t)B(t)B(t)^T\Phi(t_0, t)^T dt$$

is called the *controllability Gramian*.

A main result is the following
The state equation is controllable on \((t_0, t_f)\) if and only if the controllability Gramian \(W(t_0, t_f)\) is invertible.

Remark: We will see later (Lec.6) that the minimal (squared) control energy, defined by 
\[
\|u\|^2 := \int_{t_0}^{t_f} |u|^2 dt,
\]
needed to move from \(x(t_0) = x_0\) to \(x(t_f) = 0\) equals 
\[
x_0^T W(t_0, t_f)^{-1} x_0.
\]
Proof of Th.1

i) Suppose first $W$ is invertible. Given $x_0$ the control signal

$$u(t) = -B^T \Phi^T(t_0, t) W^{-1}(t_0, t_f)x_0$$

will give $x(t_f) = 0$ (check!). Hence the system is controllable.

ii) Suppose instead the system is controllable. Want to show $W$ invertible, i.e. that $Wx_0 = 0$ implies $x_0 = 0$.

Find $u$ so $0 = \Phi x_0 + \int \Phi Bu dt$, i.e.

$$x_0 = -\int_{t_0}^{t_f} \Phi(t_0, t)B(t)u(t)dt$$

$$x_0^T x_0 = -\int_{t_0}^{t_f} x_0^T \Phi(t_0, t)B(t)u(t)dt = z(t)$$

But this shows $x_0 = 0$ since

$$\|z(t)\|^2 = \int_{t_0}^{t_f} x_0^T \Phi(t_0, t)B(t)B^T(t)\Phi^T(t_0, t)x_0 dt = x_0^T W x_0 = 0$$
The following four conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.

(ii) $\text{rank}[B \ AB \ A^2B \ \ldots \ A^{n-1}B] = n$.

(iii) $\lambda \in \mathbb{C}, \ p^T A = \lambda p^T, \ p^T B = 0 \ \Rightarrow \ p = 0$.

(iv) $\text{rank} [\lambda I - A \ B] = n \ \forall \lambda \in \mathbb{C}$.

The conditions (iii) and (iv) are called the PBH test (Popov-Belevitch-Hautus), see p221.

Notation: $C(A, B) := [B \ AB \ A^2B \ \ldots \ A^{n-1}B]$
Suppose that $0 < q < n$ and

$$\text{rank} \begin{bmatrix} B & AB & A^2B & \ldots & A^{n-1}B \end{bmatrix} = q < n$$

Then there exists an invertible $P \in \mathbb{R}^{n \times n}$ such that

$$P^{-1}AP = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}, \quad P^{-1}B = \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix}$$

where $\hat{A}_{11}$ is $q \times q$, $\hat{B}_{11}$ is $q \times m$, and

$$\text{rank} \begin{bmatrix} \hat{B}_{11} & \hat{A}_{11}\hat{B}_{11} & \ldots & \hat{A}_{11}^{q-1}\hat{B}_{11} \end{bmatrix} = q$$
Range and Null Spaces

Range space (Image) of $M : X \rightarrow Y$:
\[
\mathcal{R}(M) = \{Mx : x \in X\} \subset Y
\]

Null space (Kernel) of $M : X \rightarrow Y$:
\[
\mathcal{N}(M) = \{x : Mx = 0\} \subset X
\]

Example:
\[
\mathcal{R}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) = \left\{\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \alpha \in \mathbb{R}\right\}
\]
\[
\mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) = \left\{\alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix} : \alpha \in \mathbb{R}\right\}
\]
Cayley-Hamilton Theorem

Let \( p(s) := \det(sI - A) \) be the char. polynomial of the square matrix \( A \), then

\[
p(A) = 0
\]

This means that \( A^n \), where \( n \) is the size of \( A \), can be written as a linear combination of \( A^k \) of lower order

\[
A^n = -a_{n-1}A^{n-1} - \ldots - a_1A - a_0I
\]
Proof Th. 3

Use the $n \times n$ matrix $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ where $P_1$ is an $n \times q$ matrix with lin. indep. columns taken from $C(A, B)$ and $P_2$ is any $n \times (n - q)$ matrix making $P$ invertible. Introduce the notation

$$P^{-1} = \begin{bmatrix} M \\ N \end{bmatrix}, \text{ then } \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} P_1 & P_2 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ 0 & I_{n-q} \end{bmatrix}.$$ Note $NP_1 = 0.$

$$\mathcal{R}(B) \subset \mathcal{R}(P_1) \Rightarrow NB = 0 \Rightarrow \hat{B} = P^{-1}B = \begin{bmatrix} M \\ N \end{bmatrix}B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$\mathcal{R}(AP_1) \subset \mathcal{R}(P_1) \Rightarrow NAP_1 = 0 \Rightarrow \hat{A} = P^{-1}AP = \begin{bmatrix} M \\ N \end{bmatrix}AP = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix}$$

$$\text{rank } C(\hat{A}_{11}, \hat{B}_1) = \text{rank } C(A, B) = q$$
(i) ⇒ (ii) If (ii) fails, then after a coordinate change as in Theorem 3, $\hat{x}_2$ is unaffected by the input, so (i) fails.

(ii) ⇒ (i) If $p^T W(t_0, t_f) p = 0$ for some $p \neq 0$, then

$$
\int_{t_0}^{t_f} p^T e^{A(t_0-t)} B B^T e^{A^T(t_0-t)} p dt = 0
$$

$$
p^T e^{A(t_0-t)} B = 0 \quad \forall t \in [t_0, t_f]
$$

Differentiation with respect to $t$ at $t = t_0$, gives

$$
p^T [B \quad AB \ldots A^{n-1} B] = 0,
$$

so (ii) fails.
Proof Th2 continued

(ii) \(\Rightarrow\) (iii) If iii fails, i.e. \(p^T A = \lambda p^T\) and \(p^T B = 0\) for \(p \neq 0\) then \(p^T [B \ A B \ldots A_{n-1} B] = 0\), so (ii) fails.

(iii) \(\Rightarrow\) (ii) If Rank\( [B \ A B \ldots A_{n-1} B] = q < n\) then let \(P\) be defined as in Theorem 3 and let \(p_2^T \hat{A}_{22} = \lambda p_2^T\) and 
\(p^T = [0 \ p_2^T] P^{-1}\). Then

\[
\begin{align*}
p^T B &= [0 \ p_2^T] \begin{bmatrix} \hat{B}_{11}^1 \\ 0 \end{bmatrix} = 0 \\
p^T A &= [0 \ p_2^T] \begin{bmatrix} \hat{A}_{11}^1 & \hat{A}_{12}^1 \\ 0 & \hat{A}_{22}^1 \end{bmatrix} P^{-1} = \lambda [0 \ p_2^T] P^{-1} = \lambda p^T
\end{align*}
\]

so (iii) fails.

(iv) \(\Leftrightarrow\) \(\{p^T [\lambda - A \ B] = 0 \Rightarrow p = 0\} \Leftrightarrow (iii)\)
Tank example - controllable?

\[ \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \]

\[ \dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \]
Tank example - controllable?

\[
\dot{x} = \begin{bmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{bmatrix} x + \begin{bmatrix}
1 & 0 \\
1 & 1 \\
0 & 1
\end{bmatrix} u
\]
Example - Single Input Diagonal Systems

For which \( \lambda_i, b_i \) is this system controllable?

\[
\dot{x} = \begin{bmatrix}
\lambda_1 & 0 \\
\lambda_2 & \ddots \\
0 & \ddots & \ddots \\
& & \ddots & \lambda_n \\
\end{bmatrix} x + \begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_n \\
\end{bmatrix} u
\]

Method 1: When is the controllability matrix invertible?

\[
C(A, B) = \begin{bmatrix}
b_1 & b_1\lambda_1 & b_1\lambda_1^2 & \ldots & b_1\lambda_1^{n-1} \\
b_2 & b_2\lambda_2 & b_2\lambda_2^2 & \ldots & b_2\lambda_2^{n-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
b_n & b_n\lambda_n & b_n\lambda_n^2 & \ldots & b_n\lambda_n^{n-1} \\
\end{bmatrix}
\]

After some work: When all \( \lambda_i \) are distinct and all \( b_i \) nonzero.

Method 2: The PBH-test gives you this result immediately!
The equation

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = 0 \]

is called \textit{reachable on} \((t_0, t_f)\), if for any \(x_f\), there exists \(u(t)\) such that \(x(t_f) = x_f\).

The matrix function

\[ W_r(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_f, t)B(t)B(t)^T \Phi(t_f, t)^T \, dt \]

\[ = \Phi(t_f, t_0)W(t_0, t_f)\Phi(t_f, t_0)^T \]

is called the \textit{reachability Gramian}.

Continuous time controllability and reachability are equivalent
The equation
\[
\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0
\]
\[
y(t) = C(t)x(t)
\]
is called *observable on* \([t_0, t_f]\) if any initial state \(x_0\) is uniquely determined by the output \(y(t)\) for \(t \in [t_0, t_f]\).

It is called *reconstructable on* \([t_0, t_f]\) if the state \(x(t_f)\) is uniquely determined by the output \(y(t)\) for \(t \in [t_0, t_f]\).

In continuous time, observability and reconstructability are equivalent (why?)
The matrix function

$$M(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t, t_0)^T C(t)^T C(t) \Phi(t, t_0) \, dt$$

is called the **observability Gramian** of the system

$$\dot{x}(t) = A(t)x(t)$$

$$y(t) = C(t)x(t)$$

Remark: Operator interpretation (see later)

$$M(t_0, t_f) = L^* L$$

where $L : \mathbb{R}^n \to L^m_2(t_0, t_f)$ with

$$(Lx_0)(t) = C(t)\Phi(t, t_0)x_0, \quad x_0 \in \mathbb{R}^n$$
Degree of Observability

The following two conditions are equivalent

(i) The system \( \{A(t), C(t)\} \) is observable on \([t_0, t_f]\).

(ii) \( M(t_0, t_f) > 0 \)

Interpretations: Consider \( y(t) = Lx_0 + e(t) \)

i) If \( e \) is white noise with unit variance then \( E|y - Lx_0|^2 \) is minimized for \( \hat{x}_0 = (L^*L)^{-1}L^*y \) and the variance of the estimate is \( (L^*L)^{-1} = M(t_0, t_f)^{-1} \).

ii) The set of \( x_0 \) for which \( \exists e(t) \) with \( \|e\|^2 \leq \sigma^2 \) such that \( y(t) \equiv 0 \) is given by

\[
x_0^T M(t_0, t_f)x_0 \leq \sigma^2
\]
The following four conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ is observable.

(ii) $\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n.$

(iii) $\lambda \in \mathbb{C}$: $Ap = \lambda p$, $Cp = 0 \Rightarrow p = 0$

(iv) $\text{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}.$
Theorem 6 - Unobservable State Equation

Suppose that \( \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = l < n \)

Then there exists an invertible \( Q \in \mathbb{R}^{n \times n} \) such that

\[
Q^{-1} A Q = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad C Q = \begin{bmatrix} \hat{C}_{11} & 0 \end{bmatrix}
\]

where \( \hat{A}_{11} \) is \( l \times l \), \( \hat{C}_{11} \) is \( p \times l \), and rank \( \begin{bmatrix} \hat{C}_{11} \\ \hat{C}_{11} \hat{A}_{11} \\ \vdots \\ \hat{C}_{11} \hat{A}_{11}^{l-1} \end{bmatrix} = l \).
Suppose \((A, b)\) is controllable. There is an invertible \(P\) such that a state transformation will bring the system to the form

\[
PAP^{-1} = A_c = \begin{bmatrix}
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 \\
-a_0 & -a_1 & \cdots & -a_{n-1}
\end{bmatrix}, \quad PB = B_c = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

\[
\det(sI - A) = s^n + a_{n-1}s^{n-1} + \cdots + a_1s + a_0
\]
Proof

Introduce some notation for $C^{-1}(A, b)$:

$$\begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} := [b \quad Ab \ldots A^{n-1}b]^{-1} \Rightarrow M_nA^k b = 0, \quad k = 0, \ldots, n - 2$$

$$M_nA^{n-1}b = 1$$

We can use the transformation $z = Px$ where

$$P = \begin{bmatrix} M_n \\ M_nA \\ \vdots \\ M_nA^{n-1} \end{bmatrix}$$

That $P$ is invertible follows from calculation of $PC$ (the new controllability matrix)
Proof

\[
PC = \begin{bmatrix}
M_n \\
M_nA \\
\vdots \\
M_nA^{n-1}
\end{bmatrix} \begin{bmatrix}
b & Ab & \ldots & A^{n-1}b
\end{bmatrix} = \begin{bmatrix}
0 & \ldots & 0 & 1 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 1 & \ast & \ast \\
1 & \ast & \ldots & \ast
\end{bmatrix}
\]

\[
PA = \begin{bmatrix}
M_nA \\
M_nA^2 \\
\vdots \\
M_nA^n
\end{bmatrix} = \begin{bmatrix}
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
-a_0 & -a_1 & \ldots & -a_{n-1}
\end{bmatrix} \begin{bmatrix}
M_n \\
M_nA \\
\vdots \\
M_nA^{n-1}
\end{bmatrix} = A_cP
\]

\[
PB = \begin{bmatrix}
M_nb \\
M_nAb \\
\vdots \\
M_nA^{n-1}b
\end{bmatrix} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
1
\end{bmatrix} = B_c
To construct the corresponding controller form when we have multiple inputs \((m > 1)\) we need the following

**Definition:** Let \(B = [B_1 \ldots B_m]\). For \(j = 1, \ldots, m\), the *controllability index* \(\rho_j\) is the smallest integer such that \(A^{\rho_j} B_j\) is linearly dependent on the column vectors occurring to the left of it in the controllability matrix

\[
\begin{bmatrix}
B & AB & \ldots & A^{n-1}B
\end{bmatrix}
\]
Given a contr. system \( \{A, B\} \), with controllability indices \( \rho_1, \ldots, \rho_m \), define

\[
M = \begin{bmatrix}
  M_1 \\
  \vdots \\
  M_n
\end{bmatrix}
:=
\begin{bmatrix}
  B_1 & AB_1 & \ldots & A^{\rho_1-1}B_1 & \ldots & B_m & \ldots & A^{\rho_m-1}B_m
\end{bmatrix}^{-1}
\]

\[
P = \begin{bmatrix}
  P_1 \\
  \vdots \\
  P_m
\end{bmatrix},
\quad P_i = \begin{bmatrix}
  M_{\rho_1+\cdots+\rho_i} \\
  M_{\rho_1+\cdots+\rho_i}A \\
  \vdots \\
  M_{\rho_1+\cdots+\rho_i}A^{\rho_i-1}
\end{bmatrix}
\]

Notice that it is rather easy to write Matlab code for this.

See Rugh 13.9 for the proof of the following result.
The transformation \( z = Px \) gives \((A_c, B_c)\) with

\[
A_c = \begin{bmatrix}
1 & \cdots & \cdots & 1 \\
\star & \cdots & \cdots & \star \\
\star & \cdots & \cdots & \star \\
\star & \cdots & \cdots & \star
\end{bmatrix}
\]
Theorem 7, Controller Form - Multiple Inputs

The block sizes equal the controllability indices $\rho_i$.

If $B$ is not full rank, $B_c$ will have a stair-case form.
Using the controller form it is now easy to prove
Suppose \((A, B)\) is controllable. Given a monic polynomial \(p(s)\) there is a feedback control \(u = -Kx\) so that

\[
\det(sI - A - BK) = p(s).
\]

**Proof** We can get rid of the \(\star\) elements in \(B_c\) by writing \(B_c = \tilde{B}_c T\) where \(T\) is an upper triangular matrix with right inverse. Introduce the new control signal \(\tilde{u} = Tu\). By state feedback we can now change each line of stars in \(A_c\). We can for instance transform \(A_c\) to a controller form with one big block, with the last row containing the coefficients of \(p(s)\).
Definition - Observability Index

Let $C^T = [C_1^T \ldots C_p^T]^T$. For $j = 1, \ldots, p$, the observability index $\eta_j$ is the smallest integer such that $C_j A^{\eta_j}$ is linearly dependent on the row vectors occurring above it in the observability matrix

$$
\begin{bmatrix}
    C \\
    CA \\
    \vdots \\
    CA^{n-1}
\end{bmatrix}
$$
Suppose \((C, A)\) is observable. Then there is a transformation 
\[ z = Px, \] 
to the form 
\[ \dot{z} = A_oz, \quad y = C_oz \] 
with

\[
A_o = \text{transpose of the form for } A_c \text{ above} \\
C_o = \text{transpose of the form for } B_c \text{ above}
\]

The size of the blocks equals the observability indices \(\eta_j\).
Theorem 9 - Time-Invariant Gramian

Let $A$ be exponentially stable. Then, the reachability Gramian $W_r(-\infty, 0)$ equals the unique solution $P$ to the matrix equation

$$PA^T + AP = -BB^T$$

Similarly, the observability Gramian $M(0, \infty)$ equals the solution $Q$ of

$$QA + A^TQ = -C^TC$$
Proof of Theorem 9

Let \( P = W_r(\mathbin{-\infty}, 0) = \int_0^\infty e^{A\sigma} B B^T e^{A^T\sigma} d\sigma \). Then

\[
PA^T + AP = \int_0^\infty \frac{\partial}{\partial \sigma} \left( e^{A\sigma} B B^T e^{A^T\sigma} \right) d\sigma \\
= \left[ e^{A\sigma} B B^T e^{A^T\sigma} \right]_0^\infty \\
= -BB^T
\]

The linear operator (Lyapunov 1893)

\[
L(P) = AP + PA^T
\]

has \( \mathcal{R}(L) = \mathbb{R}^{n \times n} \) so \( \mathcal{N}(L) = \{0\} \) and the solution \( P \) is unique.

The equation for the observability Gramian is obtained by replacing \( A, B \) with \( A^T, C^T \).
Balanced Realization

For the stable system \((A, B, C)\), with Gramians \(P\) and \(Q\), the variable transformation \(\hat{x} = Tx\) gives

\[
\hat{P} = TPT^* \\
\hat{Q} = T^{-*}QT^{-1}
\]

Choosing \(R, T\), unitary \(U\) and diagonal \(\Sigma\) from

\[
Q = R^*R \quad \text{(Choleski Factorisation)} \\
RPR^* = U\Sigma^2U^* \quad \text{(Singular Value Decomposition)} \\
T = \Sigma^{-1/2}U^*R
\]

gives (check)

\[
\hat{P} = \hat{Q} = \Sigma
\]

The corresponding realization \((\hat{A}, \hat{B}, \hat{C})\) is called a balanced realization of the system \((A, B, C)\).
Let the states be sorted such that $\Sigma$ is decreasing. The diagonal elements of $\Sigma$ measure “how controllable and observable” the corresponding states are. With

$$
\hat{A} = \begin{bmatrix}
\hat{A}_{11} & \hat{A}_{12} \\
\hat{A}_{21} & \hat{A}_{22}
\end{bmatrix}, \quad \hat{B} = \begin{bmatrix}
\hat{B}_1 \\
\hat{B}_2
\end{bmatrix}, \quad \hat{C} = \begin{bmatrix}
\hat{C}_1 & \hat{C}_2
\end{bmatrix}
$$

$$
\Sigma = \begin{bmatrix}
\Sigma_1 & 0 \\
0 & \Sigma_2
\end{bmatrix}
$$

the system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1)$ is called a truncated balanced realization of the system $(A, B, C)$.

If $\Sigma_1 >> \Sigma_2$ the truncated system is probably a good approximation. Choose either $D = 0$ or to get correct DC-gain.
Example (done with \texttt{balreal} in MATLAB)

\[ C(sI - A)^{-1}B = \frac{1 - s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1} \]

\[ \Sigma = \text{diag}\{1.98, 1.92, 0.75, 0.33, 0.15, 0.0045\} \]

\[ \hat{C}(sI - \hat{A})^{-1}\hat{B} = \frac{0.20s^2 - 0.44s + 0.23}{s^3 + 0.44s^2 + 0.66s + 0.17} \]
Bonus: Full Kalman Decomposition

Simultaneous controller and observer decomposition

Use $P = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}$ where $P_i$ has $n_i$ columns with

Columns of $\begin{bmatrix} P_1 & P_2 \end{bmatrix}$ basis for $\mathcal{R}(C)$
Columns of $P_2$ basis for $\mathcal{R}(C) \cap \mathcal{N}(O)$
Columns of $\begin{bmatrix} P_2 & P_4 \end{bmatrix}$ basis for $\mathcal{N}(O)$
Columns of $P_3$ chosen so $P$ invertible.

\[\hat{A} = \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix}, \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix}\]

\[C = \begin{bmatrix} \hat{C}_1 & 0 & \hat{C}_3 & 0 \end{bmatrix}\]
Kalman’s Decomposition Theorem

The system \((\hat{A}_{11}, \hat{B}_1, \hat{C}_1)\) is both controllable and observable. It is of minimal order, \(n_1\). The transfer function equals \(\hat{C}_1(sI - \hat{A}_1)^{-1}\hat{B}_1\).
**Bonus: More on Controllability**

\( A, B \) is controllable if and only if

- The only \( C \) for which \( C(sI - A)^{-1}B = 0, \forall s \) is \( C = 0 \)

\( A, C \) is observable if and only if

- The only \( B \) for which \( C(sI - A)^{-1}B = 0, \forall s \) is \( B = 0 \)

Proof: \( 0 = C(sI - A)^{-1}B = \sum_{k=0}^{\infty} CA^k B / s^{k+1} \iff 0 = CA^k B, \forall k \)

\[
0 = C \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \iff 0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} B
\]
Let $G_1(s) = C_1(sI - A_1)^{-1}B_1$ and $G_2(s) = C_2(sI - A_2)^{-1}B_2$

If $A_1$ and $A_2$ have no common eigenvalues then

$$G_1(s) + G_2(s) \equiv 0 \implies G_1(s) = G_2(s) = 0$$

Proof: Can assume both systems are minimal. From

$$G_1(s) + G_2(s) = [C_1 \ C_2] \begin{bmatrix} sI - A_1 & 0 \\ 0 & s - A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0$$

and the fact that $[C_1 \ C_2], \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ is observable (PBH-test),

the previous frame shows that $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Assume \((A, b, c)\) minimal and that \(z\) is not an eigenvalue of \(A\). Then the following are equivalent

- \(G(z) = c(zI - A)^{-1}b + d = 0\)
- With \(u_0\) arbitrary and \(x_0 := (zI - A)^{-1}bu_0\) we have
  \[
  \begin{bmatrix}
  zI - A & -b \\
  c & d
  \end{bmatrix}
  \begin{bmatrix}
  x_0 \\
  u_0
  \end{bmatrix} = 0
  \]
- The following matrix looses rank
  \[
  \begin{bmatrix}
  zI - A & -b \\
  c & d
  \end{bmatrix}
  \]
Given two minimal systems \( n_i(s)/d_i(s) = c_i(sI - A_i)^{-1}b_i, \quad i = 1, 2 \)

Then the series connection \( \frac{n_2(s)}{d_2(s)} \frac{n_1(s)}{d_1(s)} \) is

- uncontrollable \( \iff \) there is \( z \) so \( n_1(z) = d_2(z) = 0 \)
- unobservable \( \iff \) there is \( z \) so \( n_2(z) = d_1(z) = 0 \)

Proof:

Controllable, check when rank \( \begin{bmatrix} zI - A_1 & 0 & b_1 \\ -b_2c_1 & zI - A_2 & 0 \end{bmatrix} \leq n \)

Observable, check when rank \( \begin{bmatrix} zI - A_1 & 0 \\ -b_2c_1 & zI - A_2 \end{bmatrix} \leq n \)