Lecture 3

- Controllability
- Observability
- Controller and Observer Forms
- Balanced Realizations

Rugh, chapters 9,13, 14 (only pp 247-249) and (25)

Controllability

How should **controllability** be defined ?

Some (not used) alternatives:

By proper choice of control signal u

- any state x_0 can be made an equilibrium
- any state trajectory $x(t)$ can be obtained
- any output trajectory $y(t)$ can be obtained

The most fruitful definition has instead turned out to be the following

Controllability

The state equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0$$

is called *controllable on* (t_0, t_f), if for any x_0, there exists $u(t)$ such that $x(t_f) = 0$ ("Controllable to origin")

Question: Is this equivalent to the following definition:

"for $x_0 = 0$ and any x_1, there exists $u(t)$ such that $x(t_f) = x_1$"

("Controllable from origin")

The audience is thinking!

Hint: $x(t_f) = \Phi(t_f, t_0)x(t_0) + \int_{t_0}^{t_f} \Phi(t_f, t)B(t)u(t)dt$

Controllability Gramian

The matrix function

$$W(t_0, t_f) = \int_{t_0}^{t_f} \Phi(t_0, t) B(t) B(t)^T \Phi(t_0, t)^T dt$$

is called the *controllability Gramian*.

A main result is the following

Th.1 Controllability Criterion (Rugh 9.2)

The state equation is controllable on (t_0, t_f) if and only if the controllability Gramian $W(t_0, t_f)$ is invertible.

Remark: We will see later (Lec.6) that the minimal (squared) control energy, defined by $\|u\|^2 := \int_{t_0}^{t_f} |u|^2 dt$, needed to move from $x(t_0) = x_0$ to $x(t_f) = 0$ equals $x_0^T W(t_0, t_f)^{-1} x_0$.

Proof of Th.1

i) Suppose first W is invertible. Given x_0 the control signal

$$u(t) = -B^T \Phi^T(t_0, t) W^{-1}(t_0, t_f) x_0$$

will give $x(t_f) = 0$ (check!). Hence the system is controllable.

ii) Suppose instead the system is controllable. Want to show W invertible, i.e. that $W x_0 = 0$ implies $x_0 = 0$.

Find u so $0 = \Phi x_0 + \int \Phi B u dt$, i.e.
$x_0 = -\int_{t_0}^{t_f} \Phi(t_0, t) B(t) u(t) dt$

$$x_0^T x_0 = -\int_{t_0}^{t_f} \underbrace{x_0^T \Phi(t_0, t) B(t)}_{:= z(t)} u(t) dt$$

But this shows $x_0 = 0$ since

$$\|z(t)\|^2 = \int_{t_0}^{t_f} x_0^T \Phi(t_0, t) B(t) B^T(t) \Phi^T(t_0, t) x_0 dt = x_0^T W x_0 = 0$$

Th2. LTI Controllability Test - (Rugh 9.5)

The following four conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t) + Bu(t)$ is controllable.

(ii) $\mathrm{rank}[B\ AB\ A^2B\ \ldots\ A^{n-1}B] = n$.

(iii) $\lambda \in \mathbf{C}$, $p^T A = \lambda p^T, p^T B = 0 \quad \Rightarrow p = 0$.

(iv) rank $[\lambda I - A \quad B] = n \quad \forall \lambda \in \mathbf{C}$.

The conditions (iii) and (iv) are called the PBH test (Popov-Belevitch-Hautus), see p221.

Notation: $\mathcal{C}(A, B) := [B\ AB\ A^2B\ \ldots\ A^{n-1}B]$

Th.3 LTI Uncontrollable System Decomposition

Suppose that $0 < q < n$ and

$$\mathrm{rank}\begin{bmatrix}B & AB & A^2B & \dots & A^{n-1}B\end{bmatrix} = q < n$$

Then there exists an invertible $P \in \mathbf{R}^{n\times n}$ such that

$$P^{-1}AP = \begin{bmatrix}\widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22}\end{bmatrix}, \qquad P^{-1}B = \begin{bmatrix}\widehat{B}_{11} \\ 0\end{bmatrix}$$

where $\widehat{A}_{11}$ is $q \times q$, $\widehat{B}_{11}$ is $q \times m$, and

$$\mathrm{rank}[\widehat{B}_{11} \quad \widehat{A}_{11}\widehat{B}_{11} \dots \widehat{A}_{11}^{q-1}B_{11}] = q$$

Range and Null Spaces

Range space (Image) of $M : X \to Y$:

$$\mathcal{R}(M) \;=\; \{Mx : x \in X\} \subset Y$$

Null space (Kernal) of $M : X \to Y$:

$$\mathcal{N}(M) \;=\; \{x : Mx = 0\} \subset X$$

Example:

$$\mathcal{R}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \;=\; \left\{\alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix} : \alpha \in \mathbf{R}\right\}$$

$$\mathcal{N}\left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\right) \;=\; \left\{\alpha \begin{bmatrix} 2 \\ -1 \end{bmatrix} : \alpha \in \mathbf{R}\right\}$$

Cayley-Hamilton Theorem

Let $p(s) := \det(sI - A)$ be the char. polynomial of the square matrix A, then

$$p(A) = 0$$

This means that A^n, where n is the size of A, can be written as a linear combination of A^k of lower order

$$A^n = -a_{n-1}A^{n-1} - \ldots - a_1 A - a_0 I$$

Proof Th. 3

Use the $n \times n$ matrix $P = [P_1 \;\; P_2]$ where P_1 is an $n \times q$ matrix with lin. indep. columns taken from $\mathcal{C}(A, B)$ and P_2 is any $n \times (n-q)$ matrix making P invertible. Introduce the notation

$$P^{-1} = \begin{bmatrix} M \\ N \end{bmatrix}, \text{ then } \begin{bmatrix} M \\ N \end{bmatrix} [P_1 \;\; P_2] = \begin{bmatrix} I_q & 0 \\ 0 & I_{n-q} \end{bmatrix}. \text{ Note } NP_1 = 0.$$

$$\mathcal{R}(B) \subset \mathcal{R}(P_1) \Rightarrow NB = 0 \Rightarrow \quad \widehat{B} = P^{-1}B = \begin{bmatrix} M \\ N \end{bmatrix} B = \begin{bmatrix} \widehat{B}_1 \\ 0 \end{bmatrix}$$

$$\mathcal{R}(AP_1) \subset \mathcal{R}(P_1) \Rightarrow NAP_1 = 0 \Rightarrow \widehat{A} = P^{-1}AP = \begin{bmatrix} M \\ N \end{bmatrix} AP = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ 0 & \widehat{A}_{22} \end{bmatrix}$$

$$\operatorname{rank} \mathcal{C}(\widehat{A}_{11}, \widehat{B}_1) = \operatorname{rank} \mathcal{C}(A, B) = q$$

Proof of Th. 2

(i) $\Rightarrow$ (ii) If (ii) fails, then after a coordinate change as in Theorem 3, $\hat{x}_2$ is unaffected by the input, so (i) fails.

(ii) $\Rightarrow$ (i) If $p^T W(t_0, t_f)p = 0$ for some $p \neq 0$, then

$$\int_{t_0}^{t_f} p^T e^{A(t_0-t)} B B^T e^{A^T(t_0-t)} p dt = 0$$

$$p^T e^{A(t_0-t)} B = 0 \quad \forall t \in [t_0, t_f]$$

Differentiation with respect to t at $t = t_0$, gives

$$p^T [B \quad AB \dots A^{n-1} B] = 0,$$

so (ii) fails.

Proof Th2 continued

(ii) $\Rightarrow$ **(iii)** If iii fails, i.e. $p^T A = \lambda p^T$ and $p^T B = 0$ for $p \neq 0$ then $p^T[B \quad AB \ldots A^{n-1}B] = 0$, so **(ii)** fails.

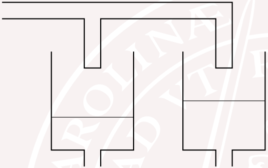
(iii) $\Rightarrow$ **(ii)** If $\mathrm{rank}[B \ \ \ldots \ \ A^{n-1}B] = q < n$ then let P be defined as in Theorem 3 and let ${p_2}^T \hat{A}_{22} = \lambda {p_2}^T$ and $p^T = [0 \quad {p_2}^T]P^{-1}$. Then

$$
\begin{aligned}
p^T B &= [0 \quad {p_2}^T] \begin{bmatrix} \hat{B}_{11} \\ 0 \end{bmatrix} = 0 \\
p^T A &= [0 \quad {p_2}^T] \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{22} \end{bmatrix} P^{-1} = \lambda [0 \quad {p_2}^T] P^{-1} = \lambda p^T
\end{aligned}
$$

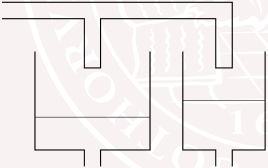
so **(iii)** fails.

(iv) $\Leftrightarrow \left\{ p^T[\lambda - A \quad B] = 0 \ \Rightarrow \ p = 0 \right\} \Leftrightarrow$ **(iii)**

Tank example - controllable?

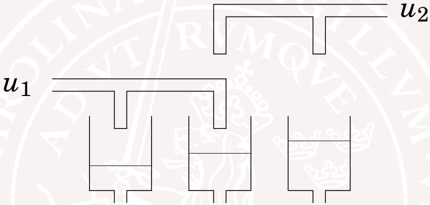


$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$



$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

Tank example - controllable?



$$\dot{x} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} u$$

Example - Single Input Diagonal Systems

For which λ_i, b_i is this system controllable?

$$\dot{x} = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} x + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} u$$

Method 1: When is the controllability matrix invertible?

$$\mathcal{C}(A,B) = \begin{bmatrix} b_1 & b_1\lambda_1 & b_1\lambda_1^2 & \dots b_1\lambda_1^{n-1} \\ b_2 & b_2\lambda_2 & b_2\lambda_2^2 & \dots b_2\lambda_2^{n-1} \\ \vdots & & & \\ b_n & b_n\lambda_n & b_n\lambda_n^2 & \dots b_n\lambda_n^{n-1} \end{bmatrix}$$

After some work: When all λ_i are distinct and all b_i nonzero.

Method 2: The PBH-test gives you this result immediately!

LTV Reachability

The equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = 0$$

is called *reachable on* (t_0, t_f), if for any x_f, there exists $u(t)$ such that $x(t_f) = x_f$.

The matrix function

$$\begin{aligned} W_r(t_0, t_f) &= \int_{t_0}^{t_f} \Phi(t_f, t)B(t)B(t)^T\Phi(t_f, t)^T dt \\ &= \Phi(t_f, t_0)W(t_0, t_f)\Phi(t_f, t_0)^T \end{aligned}$$

is called the *reachability Gramian*.

Continuous time controllability and reachability are equivalent

LTV Observability

The equation

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$
$$y(t) = C(t)x(t)$$

is called *observable on* $[t_0, t_f]$ if any initial state x_0 is uniquely determined by the output $y(t)$ for $t \in [t_0, t_f]$.

It is called *reconstructable on* $[t_0, t_f]$ if the state $x(t_f)$ is uniquely determined by the output $y(t)$ for $t \in [t_0, t_f]$.

In continuous time, observability and reconstrubality are equivalent (why?)

Observability Gramian

The matrix function

$$M(t_0,t_f) = \int_{t_0}^{t_f} \Phi(t,t_0)^T C(t)^T C(t) \Phi(t,t_0) dt$$

is called the *observability Gramian* of the system

$$\dot{x}(t) = A(t)x(t)$$
$$y(t) = C(t)x(t)$$

Remark: Operator interpretation (see later)

$$M(t_0,t_f) = L^* L$$

where $L : \mathbf{R}^n \to L_2^m(t_0,t_f)$ with

$$(Lx_0)(t) = C(t)\Phi(t,t_0)x_0, \quad x_0 \in \mathbf{R}^n$$

Degree of Observability

The following two conditions are equivalent

(i) The system $\{A(t), C(t)\}$ is observable on $[t_0, t_f]$.

(ii) $M(t_0, t_f) > 0$

Interpretations: Consider $y(t) = Lx_0 + e(t)$

i) If e is white noise with unit variance then $E|y - Lx_0|^2$ is minimized for $\hat{x}^0 = (L^*L)^{-1}L^*y$ and the variance of the estimate is $(L^*L)^{-1} = M(t_0, t_f)^{-1}$.

ii) The set of x_0 for which $\exists\ e(t)$ with $\|e\|^2 \leq \sigma^2$ such that $y(t) \equiv 0$ is given by

$$x_0^T M(t_0, t_f) x_0 \leq \sigma^2$$

Th. 5 (Rugh 9.11) - LTI Observability

The following four conditions are equivalent:

(i) The system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ is observable.

(ii) $\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$.

(iii) $\lambda \in \mathbf{C}: \; Ap = \lambda p, \, Cp = 0 \quad \Rightarrow p = 0$

(iv) $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbf{C}$.

Theorem 6 - Unobservable State Equation

Suppose that $\operatorname{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = l < n$

Then there exists an invertible $Q \in \mathbf{R}^{n \times n}$ such that

$$Q^{-1}AQ = \begin{bmatrix} \hat{A}_{11} & 0 \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \qquad CQ = \begin{bmatrix} \hat{C}_{11} & 0 \end{bmatrix}$$

where $\hat{A}_{11}$ is $l \times l$, $\hat{C}_{11}$ is $p \times l$, and $\operatorname{rank} \begin{bmatrix} \hat{C}_{11} \\ \hat{C}_{11}\hat{A}_{11} \\ \vdots \\ \hat{C}_{11}\hat{A}_{11}^{l-1} \end{bmatrix} = l$.

LTI Controller Canonical Form - Single Input

Suppose (A, b) is controllable. There is an invertible P such that a state transformation will bring the system to the form

$$PAP^{-1} = A_c = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix}, \qquad PB = B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

Proof

Introduce some notation for $\mathcal{C}^{-1}(A,b)$:

$$\begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} := \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix}^{-1} \Rightarrow \begin{array}{l} M_n A^k b = 0, \quad k = 0, \dots, n-2 \\ M_n A^{n-1} b = 1 \end{array}$$

We can use the transformation $z = Px$ where

$$P = \begin{bmatrix} M_n \\ M_n A \\ \vdots \\ M_n A^{n-1} \end{bmatrix}$$

That P is invertible follows from calculation of $P\mathcal{C}$ (the new controllability matrix)

Proof

$$PC = \begin{bmatrix} M_n \\ M_nA \\ \vdots \\ M_nA^{n-1} \end{bmatrix} \begin{bmatrix} b & Ab & \dots & A^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \vdots & \iddots & \star \\ 0 & 1 & \star & \star \\ 1 & \star & \dots & \star \end{bmatrix}$$

$$PA = \begin{bmatrix} M_nA \\ M_nA^2 \\ \vdots \\ M_nA^n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} M_n \\ M_nA \\ \vdots \\ M_nA^{n-1} \end{bmatrix} = A_cP$$

$$PB = \begin{bmatrix} M_nb \\ M_nAb \\ \vdots \\ M_nA^{n-1}b \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = B_c$$

Controllability Index

To construct the corresponding controller form when we have multiple inputs ($m > 1$) we need the following

Definition: Let $B = [B_1 \ \ldots \ B_m]$. For $j = 1, \ldots, m$, the *controllability index* ρ_j is the smallest integer such that $A^{\rho_j} B_j$ is linearly dependent on the column vectors occuring to the left of it in the controllability matrix

$$\begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$$

Notation for Controller Form

Given a contr. system $\{A, B\}$, with controllability indices $\rho_1, \ldots \rho_m$, define

$$M = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix} := \begin{bmatrix} B_1 & AB_1 \ldots A^{\rho_1 - 1} B_1 & \ldots & B_m \ldots A^{\rho_m - 1} B_m \end{bmatrix}^{-1}$$

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_m \end{bmatrix}, \quad P_i = \begin{bmatrix} M_{\rho_1 + \cdots + \rho_i} \\ M_{\rho_1 + \cdots + \rho_i} A \\ \vdots \\ M_{\rho_1 + \cdots + \rho_i} A^{\rho_i - 1} \end{bmatrix}$$

Notice that it is rather easy to write Matlab code for this.

See Rugh 13.9 for the proof of the following result

The transformation $z = Px$ gives (A_c, B_c) with

$$
A_c = \left[\begin{array}{cccc|cccc|cccc}
 & 1 & & & & & & & & & & \\
 & & \ddots & & & & & & & & & \\
 & & & 1 & & & & & & & & \\
\star & \dots & \dots & \star & \star & \dots & \dots & \star & \star & \dots & \dots & \star \\ \hline
 & & & & & & & & & & & \\
 & & & & & & & & & & & \\ \hline
 & & & & & & & & & 1 & & \\
 & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & 1 \\
\star & \dots & \dots & \star & \star & \dots & \dots & \star & \star & \dots & \dots & \star
\end{array}\right]
$$

Theorem 7, Controller Form - Multiple Inputs

$$B_c = \begin{bmatrix} \\ \hline 1 & \star & \dots & \star \\ \hline \\ \\ 0 & 1 & \star & \star \\ \hline \\ \\ \\ \hline \\ \\ 0 & \dots & 0 & 1 \end{bmatrix}$$

The block sizes equal the controllability indices ρ_i.

If B is not full rank, B_c will have a stair-case form.

LTI Feedback & Eigenvalue Assignment (Rugh 14.9)

Using the controller form it is now easy to prove

Suppose (A, B) is controllable. Given a monic polynomial $p(s)$ there is a feedback control $u = -Kx$ so that

$$\det(sI - A - BK) = p(s).$$

Proof We can get rid of the $\star$ elements in B_c by writing $B_c = \tilde{B}_c T$ where T is an upper triangular matrix with right inverse. Introduce the new control signal $\tilde{u} = Tu$. By state feedback we can now change each line of stars in A_c. We can for instance transform A_c to a controller form with one big block, with the last row containing the coefficients of $p(s)$.

Definition - Observability Index

Let $C^T = [{C_1}^T \; \dots \; {C_p}^T]^T$. For $j = 1, \dots, p$, the *observability index* η_j is the smallest integer such that $C_j A^{\eta_j}$ is linearly dependent on the row vectors occuring above it in the observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Theorem 8 -Observer form

Suppose (C, A) is observable. Then there is a transformation $z = Px$, to the form $\dot{z} = A_o z, \ \ y = C_o z$ with

$$A_o = \text{ transpose of the form for } A_c \text{ above}$$
$$C_o = \text{ transpose of the form for } B_c \text{ above}$$

The size of the blocks equals the observability indices η_j.

Theorem 9 - Time-Invariant Gramian

Let A be exponentially stable. Then, the reachability Gramian $W_r(-\infty, 0)$ equals the unique solution P to the matrix equation

$$PA^T + AP = -BB^T$$

Similarly, the observability Gramian $M(0, \infty)$ equals the solution Q of

$$QA + A^TQ = -C^TC$$

Proof of Theorem 9

Let $P = W_r(-\infty, 0) = \int_0^\infty e^{A\sigma} B B^T e^{A^T\sigma} d\sigma$. Then

$$\begin{aligned} PA^T + AP &= \int_0^\infty \frac{\partial}{\partial\sigma} \left(e^{A\sigma} B B^T e^{A^T\sigma} \right) d\sigma \\ &= \left[e^{A\sigma} B B^T e^{A^T\sigma} \right]_0^\infty \\ &= -BB^T \end{aligned}$$

The linear operator (Lyapunov 1893)

$$L(P) = AP + PA^T$$

has $\mathcal{R}(L) = \mathbf{R}^{n\times n}$ so $\mathcal{N}(L) = \{0\}$ and the solution P is unique.

The equation for the observability Gramian is obtained by replacing A, B with A^T, C^T.

Balanced Realization

For the stable system (A, B, C), with Gramians P and Q, the variable transformation $\hat{x} = Tx$ gives

$$\begin{aligned} \hat{P} &= TPT^* \\ \hat{Q} &= T^{-*}QT^{-1} \end{aligned}$$

Choosing R, T, unitary U and diagonal Σ from

$$\begin{aligned} Q &= R^*R \quad \text{(Choleski Factorisation)} \\ RPR^* &= U\Sigma^2U^* \quad \text{(Singular Value Decomposition)} \\ T &= \Sigma^{-1/2}U^*R \end{aligned}$$

gives (check)

$$\hat{P} = \hat{Q} = \Sigma$$

The corresponding realization $(\hat{A}, \hat{B}, \hat{C})$ is called a *balanced realization* of the system (A, B, C).

Truncated Balanced Realization

Let the states be sorted such that Σ is decreasing. The diagonal elements of Σ measure “how controllable and observable” the corresponding states are. With

$$\widehat{A} = \begin{bmatrix} \widehat{A}_{11} & \widehat{A}_{12} \\ \widehat{A}_{21} & \widehat{A}_{22} \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} \widehat{B}_1 \\ \widehat{B}_2 \end{bmatrix} \quad \widehat{C} = \begin{bmatrix} \widehat{C}_1 & \widehat{C}_2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

the system $(\widehat{A}_{11}, \widehat{B}_1, \widehat{C}_1)$ is called a *truncated balanced realization* of the system (A, B, C).

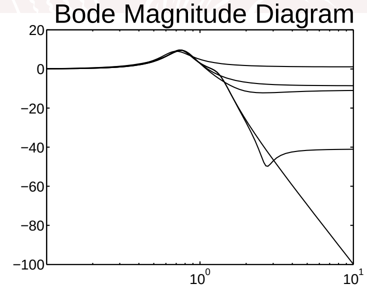
If $\Sigma_1 >> \Sigma_2$ the truncated system is probably a good approximation. Choose either $D = 0$ or to get correct DC-gain.

Example (done with `balreal` in MATLAB)

$$C(sI - A)^{-1}B = \frac{1 - s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}$$

$$\Sigma = \text{diag}\{1.98, 1.92, 0.75, 0.33, 0.15, 0.0045\}$$

$$\widehat{C}(sI - \widehat{A})^{-1}\widehat{B} = \frac{0.20s^2 - 0.44s + 0.23}{s^3 + 0.44s^2 + 0.66s + 0.17}$$



Bonus: Full Kalman Decomposition

Simultaneous controller and observer decomposition

Use $P = \begin{bmatrix} P_1 & P_2 & P_3 & P_4 \end{bmatrix}$ where P_i has n_i columns with

Columns of $\begin{bmatrix} P_1 & P_2 \end{bmatrix}$ basis for $\mathcal{R}(C)$
Columns of P_2 basis for $\mathcal{R}(C) \cap \mathcal{N}(O)$
Columns of $\begin{bmatrix} P_2 & P_4 \end{bmatrix}$ basis for $\mathcal{N}(O)$
Columns of P_3 chosen so P invertible.

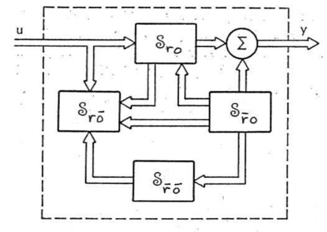
$$\begin{aligned} \hat{A} &= \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix}, \hat{B} = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \\ 0 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} \hat{C}_1 & 0 & \hat{C}_3 & 0 \end{bmatrix} \end{aligned}$$

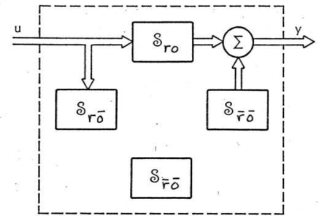
Kalman's Decomposition Theorem

The system $(\hat{A}_{11}, \hat{B}_1, \hat{C}_1)$ is both controllable and observable.

It is of minimal order, n_1

The transfer function equals $\hat{C}_1(sI - \hat{A}_1)^{-1}\hat{B}_1$.





Bonus: More on Controllability

A, B is controllable if and only if

- The only C for which $C(sI - A)^{-1}B = 0, \forall s$ is $C = 0$

A, C is observable if and only if

- The only B for which $C(sI - A)^{-1}B = 0, \forall s$ is $B = 0$

$$\text{Proof: } 0 = C(sI - A)^{-1}B = \sum_{k=0}^{\infty} CA^kB/s^{k+1} \Leftrightarrow 0 = CA^kB, \ \forall k \ \Leftrightarrow$$

$$0 = C\begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix} \quad \Leftrightarrow \quad 0 = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} B$$

Bonus: Parallel Systems

Let $G_1(s) = C_1(sI - A_1)^{-1}B_1$ and $G_2(s) = C_2(sI - A_2)^{-1}B_2$

If A_1 and A_2 have no common eigenvalues then

$$G_1(s) + G_2(s) \equiv 0 \Longrightarrow G_1(s) = G_2(s) = 0$$

Proof: Can assume both systems are minimal. From

$$G_1(s) + G_2(s) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} sI - A_1 & 0 \\ 0 & s - A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = 0$$

and the fact that $\begin{bmatrix} C_1 & C_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$ is observable (PBH-test),

the previous frame shows that $\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Bonus: System Zeros (SISO)

Assume (A, b, c) minimal and that z is not an eigenvalue of A.

Then the following are equivalent

- $G(z) = c(zI - A)^{-1}b + d = 0$
- With u_0 arbitrary and $x_0 := (zI - A)^{-1}bu_0$ we have

$$\begin{bmatrix} zI - A & -b \\ c & d \end{bmatrix} \begin{bmatrix} x_0 \\ u_0 \end{bmatrix} = 0$$

- The following matrix looses rank

$$\begin{bmatrix} zI - A & -b \\ c & d \end{bmatrix}$$

Bonus: Series Connection SISO

Given two minimal systems $n_i(s)/d_i(s) = c_i(sI - A_i)^{-1}b_i$, $i = 1, 2$

Then the series connection $\frac{n_2(s)}{d_2(s)}\frac{n_1(s)}{d_1(s)}$ is

- uncontrollable $\iff$ there is z so $n_1(z) = d_2(z) = 0$
- unobservable $\iff$ there is z so $n_2(z) = d_1(z) = 0$

Proof:

Controllable, check when $\mathrm{rank} \begin{bmatrix} zI - A_1 & 0 & b_1 \\ -b_2c_1 & zI - A_2 & 0 \end{bmatrix} \leq n$

Observable, check when $\mathrm{rank} \begin{bmatrix} zI - A_1 & 0 \\ -b_2c_1 & zI - A_2 \\ 0 & c_2 \end{bmatrix} \leq n$