Linear Systems, 2016 - Lecture 1

Lecture 1

- State equations
- Linearization
- Examples
- Transition matrices

Rugh, chapters 1-4

Main news:

- Linearization around trajectory
- Transition matrix $\Phi(t, \tau)$

Linear Time-Invariant (LTI) System

State Representation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \qquad x(0) = 0 \\ y(t) &= Cx(t) + Du(t) \end{aligned}$$

Convolution Representation

$$\begin{aligned} y(t) &= \int_0^t G(t-\tau)u(\tau)d\tau \\ G(t) &= Ce^{At}B + \delta(t)D \quad \text{(impulse response)} \end{aligned}$$

Transfer Function Representation

$$\begin{aligned} \mathbf{y}(s) &= \mathbf{G}(s)\mathbf{u}(s) \\ \mathbf{G}(s) &:= \int_{0-}^{\infty} e^{-st}G(t)dt = C(sI-A)^{-1}B + D \end{aligned}$$

Time-varying Linear System

State Representation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \qquad x(0) = 0 \\ y(t) &= C(t)x(t) + D(t)u(t) \end{aligned}$$

Integral Representation

$$y(t) = \int_0^t G(t,\tau)u(\tau)d\tau + D(t)u(t)$$

Operator Representation

$$y = Lu$$

Example: Two Tank System

Flow: $q(t)$

Volumes: V_1, V_2 (constant)

Concentrations: $u(t)$, $x_1(t)$, $x_2(t)$

Dynamics:

$$\begin{cases} \frac{d}{dt}(V_1 x_1) & = & qu - qx_1 \\ \frac{d}{dt}(V_2 x_2) & = & qx_1 - qx_2 \end{cases}$$

$$\dot{x}(t) = \begin{bmatrix} -\frac{1}{V_1} & 0 \\ \frac{1}{V_2} & -\frac{1}{V_2} \end{bmatrix} q(t)x(t) + \begin{bmatrix} \frac{1}{V_1} \\ 0 \end{bmatrix} q(t)u(t)$$

Example: Electric Circuit (RLC circuit)

See Fig 2.4

Capacitor Dynamics:

$$i(t) = \frac{d}{dt}\left(c(t)u_c(t)\right)$$

Inductor Dynamics:

$$u_l(t) = \frac{d}{dt}\left(l(t)i(t)\right)$$

State Representation: $x = [u_c \; i]^T$

$$\dot{x}(t) = \begin{bmatrix} -\dot{c}/c & 1/c \\ -1/l & -\left(r + \dot{l}\right)/l \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1/l \end{bmatrix} u(t)$$

Discrete Time LTI System

State Representation

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \qquad x(0) = 0 \\ y(k) &= Cx(k) + Du(k) \end{aligned}$$

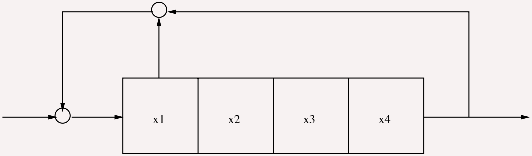
Convolution Representation

$$\begin{aligned} y(k) &= \sum_{l=0}^{k} G(k-l)u(l) \\ G(k) &= \begin{cases} D & k = 0 \\ CA^{k-1}B & k \geq 1 \end{cases} \qquad \text{(impulse response)} \end{aligned}$$

Transfer Function Representation

$$\begin{aligned} \mathbf{y}(z) &= \mathbf{G}(z)\mathbf{u}(z) \\ \mathbf{G}(z) &:= \sum_{k=0}^{\infty} G(k)z^{-k} = C(zI - A)^{-1}B + D \end{aligned}$$

Example: Shift Register



$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T$$

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x(k)$$

Linearization around a trajectory

Consider

$$\dot{x}(t) \quad = \quad f(x(t), u(t), t), \quad x(0) = x_0$$

with solution $\tilde{x}(t)$ for $u(t) = \tilde{u}(t)$ and $x_0 = \tilde{x}_0$.

Let $x_\delta = x - \tilde{x}$. Assuming differentiability of f,

$$\begin{aligned} & f(\tilde{x} + x_\delta, \tilde{u} + u_\delta, t) - f(\tilde{x}, \tilde{u}, t) \\ & = \frac{\partial f}{\partial x}(\tilde{x}, \tilde{u}, t) x_\delta + \frac{\partial f}{\partial u}(\tilde{x}, \tilde{u}, t) u_\delta + o(|x_\delta|, |u_\delta|) \end{aligned}$$

Hence, with

$$A(t) = \frac{\partial f}{\partial x}(\tilde{x}, \tilde{u}, t), \quad B(t) = \frac{\partial f}{\partial u}(\tilde{x}, \tilde{u}, t)$$

the linearization around $(\tilde{x}(t), \tilde{u}(t))$ is

$$\dot{x}_\delta(t) \quad = \quad A(t) x_\delta(t) + B(t) u_\delta(t), \quad x_\delta(0) = x_0 - \tilde{x}_0$$

Example: Communications Satellite

Spherical coordinates: $x = [r \; \dot{r} \; \theta \; \dot{\theta} \; \phi \; \dot{\phi}]^T$

Input: $u = [u_r \; u_\theta \; u_\phi]^T$, Output: $y = [r \; \theta \; \phi]^T$

Dynamics:

$$
\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), t) \\
&= \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2 \cos^2\phi + r\phi^2 - k/r^2 + u_r/m \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + 2\dot{\theta}\dot{\phi}\sin\phi/\cos\phi + u_\theta \cos\phi/(mr) \\ \dot{\phi} \\ -\dot{\theta}^2 \cos\phi \sin\phi - 2\dot{r}\dot{\phi}/r + u_\phi/(mr) \end{bmatrix}
\end{aligned}
$$

Linearized Communications Satellite

Circular equatorial orbit:

$$\begin{aligned} \tilde{x} &= \begin{bmatrix} \tilde{r} & 0 & \tilde{\omega}t & \tilde{\omega} & 0 & 0 \end{bmatrix}^T \\ \tilde{u} &\equiv 0 \end{aligned}$$

Linearization: $\dot{x} = Ax + Bu$ with

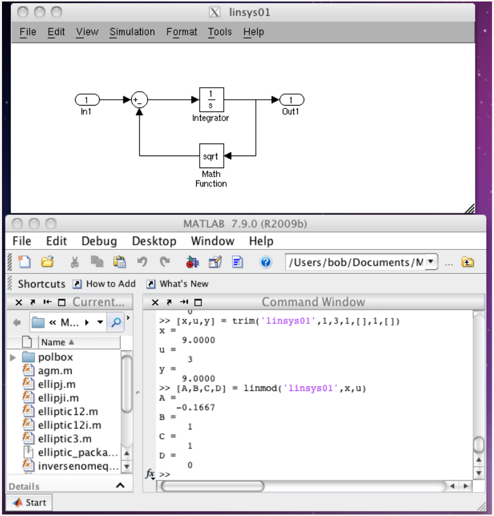
$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \tilde{\omega}^2 - \frac{2k}{\tilde{\omega}^3} & 0 & 0 & 2\tilde{\omega}\tilde{r} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2\tilde{\omega}/\tilde{r} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -\tilde{\omega}^2 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 0 & 0 & 0 \\ 1/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/(m\tilde{r}) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/(m\tilde{r}) \end{bmatrix} \end{aligned}$$

Linearization in Matlab/Simulink

```
[X,U,Y,DX]=TRIM('SYS',X0,U0,Y0,IX,IU,IY)
  fixes X, U and Y to  X0(IX), U0(IU) and Y0(IY).
  The variables IX, IU and IY are vectors of indices.

[A,B,C,D]=LINMOD('SYS',X,U) allows the state vector, X,
  and input, U, to be specified. A linear model will
  then be obtained at this operating point.
```

Linearization in Matlab/Simulink



LTV Systems - Fundamental Matrix

Can we find a counter-part to the exponential matrix

$$\Phi(t) = e^{tA}$$

for linear time-varying systems?

What properties of the LTI case carry over to LTV systems?

Discrete Time Systems

Given a matrix sequence $A(0), A(1), \ldots$ the equation

$$x(k+1) \;=\; A(k)x(k), \quad x(k_0) = x_0$$

has the unique solution

$$x(k) \;=\; \Phi(k, k_0)x_0$$

defined by the *transition matrix*

$$\Phi(k, k_0) \;=\; \begin{cases} A(k-1)\cdots A(k_0), & k > k_0 \\ I, \quad k = k_0 \end{cases}$$

Proof by inspection.

What about continuous time?

Continuous Time-varying Linear Systems

$$\dot{x}(t) \;=\; A(t)x(t), \quad x(t_0) = x_0$$

is equivalent to the integral equation

$$x(t) \;=\; x_0 + \int_{t_0}^{t} A(s)x(s)ds$$

Under weak conditions on $A(t)$ one can show convergence of

$$x_{k+1}(t) := x_0 + \int_{t_0}^{t} A(s)x_k(s)ds$$

$A(t)$ locally integrable (for instance bounded) is sufficient for existence and uniqueness

From the integral equation it is easy to see that the solution $x(t)$ depends linearly on $x(t_0)$ (how?)

Continuous Time Systems

For bounded $A(t)$, the equation

$$\dot{x}(t) = A(t)x(t), \quad x(t_0) = x_0$$

hence has a unique solution of the form

$$x(t) = \Phi(t, t_0)x_0$$

The *transition matrix* can be written as the infinite sum

$$\begin{aligned}
\Phi(t, t_0) &= I + \int_{t_0}^{t} A(\sigma_1) d\sigma_1 \\
&\quad + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) d\sigma_2 d\sigma_1 \\
&\quad + \int_{t_0}^{t} A(\sigma_1) \int_{t_0}^{\sigma_1} A(\sigma_2) \int_{t_0}^{\sigma_2} A(\sigma_3) d\sigma_3 d\sigma_2 d\sigma_1 \\
&\quad \ldots
\end{aligned}$$

Example:Time-invariant System

For

$$\dot{x} = Ax(t), \quad x(t_0) = x_0$$

the transition matrix is

$$\begin{aligned}
&\Phi(t, t_0) \\
&= I + \int_{t_0}^{t} A d\sigma_1 + \int_{t_0}^{t} A \int_{t_0}^{\sigma_1} A d\sigma_2 d\sigma_1 + \cdots \\
&= I + A(t - t_0) + A^2 \frac{(t - t_0)^2}{2} + A^3 \frac{(t - t_0)^3}{6} + \cdots \\
&= e^{A(t - t_0)}
\end{aligned}$$

so the solution is

$$x(t) = e^{A(t - t_0)} x_0$$

WARNING - Common Mistakes

If $A(t)$ is time-varying, then in general

$$\Phi(t, t_0) \neq \exp\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$$

Also beware that in general

$$e^{(A+B)t} \neq e^{At}e^{Bt}$$

Exception: If $AB = BA$ then $e^{(A+B)t} = e^{At}e^{Bt}$ holds (exercise)

Calculation of $exp(At)$ by Jordan Form

From Matrix Theory: Transformation P exist so $A = PJP^{-1}$ where J is a block diagonal matrix, each block being of the form

$$\lambda I + N = \begin{bmatrix} \lambda & 1 & 0 & \dots \\ 0 & \lambda & 1 & \dots \\ & & \ddots & 1 \\ 0 & \dots & 0 & \lambda \end{bmatrix}$$

Therefore $e^{At} = Pe^{Jt}P^{-1}$ where e^{Jt} is a block diagonal matrix, each block having form

$$e^{(\lambda I+N)t} = e^{\lambda t}e^{Nt} = e^{\lambda t}\sum_k \frac{t^k}{k!}N^k = \begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{t^2}{2!}e^{\lambda t}\dots & \\ 0 & e^{\lambda t} & te^{\lambda t} & \ddots \\ & & \ddots & \\ 0 & \dots & 0 & e^{\lambda t} \end{bmatrix}$$

Nice Example: Scalar Time-variation

Consider

$$\dot{x} = Aa(t)x(t)$$

The transition matrix is

$$\begin{aligned}\Phi(t,t_0) &= I + A\int_{t_0}^{t} a(\sigma_1)d\sigma_1 + A^2\int_{t_0}^{t} a(\sigma_1)\int_{t_0}^{\sigma_1} a(\sigma_2)d\sigma_2 d\sigma_1 + \cdots \\ &= \sum_{k=0}^{\infty}\frac{1}{k!}A^k\left[\int_{t_0}^{t} a(\sigma)d\sigma\right]^k \\ &= \exp\left(A\int_{t_0}^{t} a(\sigma)d\sigma\right)\end{aligned}$$

Second equality is nontrivial.

(Recall Two Tank Example with time-varying flow $q(t)$)

More general case: Commutating $A(t)$

If

$$A(t)\int_{t_0}^{t} A(\sigma)d\sigma = \int_{t_0}^{t} A(\sigma)d\sigma A(t)$$

then

$$\Phi(t, t_0) = \text{exp}\left\{\int_{t_0}^{t} A(\sigma)d\sigma\right\}$$

Special case: $A(t)A(\tau) = A(\tau)A(t)$ for all t, τ

Example

If $A(t) = a_1(t)A_1 + a_2(t)A_2$ where A_1 and A_2 commute then

$$\begin{aligned}\Phi(t, t_0) &= \text{exp}\left\{\int_{t_0}^{t} a_1(t)A_1 + a_2(t)A_2 dt\right\} \\ &= \text{exp}\left\{\int_{t_0}^{t} a_1(t)dtA_1\right\}\text{exp}\left\{\int_{t_0}^{t} a_2(t)dtA_2\right\}\end{aligned}$$

Characterization of $\Phi(t, t_0)$

The unique solution of the equation

$$\begin{aligned} \frac{d}{dt}X(t) &= A(t)X(t) \\ X(t_0) &= I \end{aligned}$$

is $X(t) = \Phi(t, t_0)$.

Proof. Let $x(t) = X(t)x_0$. Then

$$\dot{x}(t) = \frac{d}{dt}X(t)x_0 = A(t)X(t)x_0 = A(t)x(t)$$

so

$$x(t) = \Phi(t, t_0)x_0$$

Hence $\Phi(t, t_0)x_0 = X(t)x_0$ for every x_0, so $\Phi(t, t_0) = X(t)$

Example

$$\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 1 & \cos t \\ 0 & 0 \end{bmatrix} x(t) \\
x_2(t) &\equiv x_2(\tau) \\
\dot{x}_1(t) &= x_1(t) + \cos t \cdot x_2(\tau) \\
x_1(t) &= e^{t-\tau} x_1(\tau) + \int_\tau^t e^{t-\sigma} \cos\sigma d\sigma \cdot x_2(\tau) \\
&= e^{t-\tau} x_1(\tau) + \frac{1}{2}\Big(\sin t - \cos t - e^{t-\tau}(\sin\tau - \cos\tau)\Big) \cdot x_2(\tau) \\
\Phi(t,\tau) &= \begin{bmatrix} e^{t-\tau} & \frac{1}{2}\left(\sin t - \cos t - e^{t-\tau}(\sin\tau - \cos\tau)\right) \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Sanity check: $\Phi(t,t) = I$ and $\frac{d}{dt}\Phi(t,\tau)\Big|_{t=\tau} = \begin{bmatrix} 1 & \cos t \\ 0 & 0 \end{bmatrix}$

Input-driven Continuous System

The equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) \\ x(t_0) &= x_0 \end{aligned}$$

has the unique solution

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \sigma)B(\sigma)u(\sigma)d\sigma$$

Proof: Differentiate!

Properties of $\Phi(t, \sigma)$

For any t, τ, σ, the transition matrix satisfies

$$\begin{aligned}
\Phi(t,\tau) &= \Phi(t,\sigma)\Phi(\sigma,\tau) \text{ (semigroup property)} \\
\frac{d}{dt}\Phi(t,\sigma) &= A(t)\Phi(t,\sigma) \\
\frac{d}{d\sigma}\Phi(t,\sigma) &= -\Phi(t,\sigma)A(\sigma)
\end{aligned}$$

Proof of first property: Let $R(t) = \Phi(t,\sigma)\Phi(\sigma,\tau)$. Then

$$\begin{aligned}
\frac{d}{dt}R(t) &= A(t)R(t) \\
R(\sigma) &= \Phi(\sigma,\tau)
\end{aligned}$$

so $R(t)$ must be identical to $\Phi(t,\tau)$

Properties of $\Phi(t, \sigma)$

Proof of third property:

$$\Phi(\sigma + h, \sigma) = I + hA(\sigma) + o(h) \qquad \text{(why?)}$$

Hence, using first property, we have

$$\Phi(t, \sigma) = \Phi(t, \sigma + h)(I + hA(\sigma) + o(h))$$

from which we get

$$\frac{1}{h}(\Phi(t, \sigma + h) - \Phi(t, \sigma)) = -\Phi(t, \sigma + h)A(\sigma) + o(1)$$

from which the result follows as $h \to 0$

$$\frac{d}{d\sigma}\Phi(t, \sigma) \quad = \quad -\Phi(t, \sigma)A(\sigma)$$

Inversion

The transition matrix $\Phi(t, t_0)$ is invertible for any t, t_0 and

$$\Phi(t, t_0)^{-1} \quad = \quad \Phi(t_0, t)$$

Proof. By the composition rule

$$\Phi(t, t_0)\Phi(t_0, t) = \Phi(t_0, t)\Phi(t, t_0) = \Phi(t_0, t_0) = I$$

Warning: Stability is NOT determined by eigenvalues

Stability for a time-varying system

$$\dot{x} = A(t)x$$

can NOT be determined by the eigenvalues of $A(t)$

For stability, location of the eigenvalues

$$\lambda(A(t))$$

in the left half plane for all t is neither sufficient or necessary!

Try to figure out a counter-example yourself!

(There will be one in Lecture 2)