5

Stability and Performance of Feedback Systems

This chapter introduces the feedback structure and discusses its stability and performance properties. The arrangement of this chapter is as follows: Section 5.1 discusses the necessity for introducing feedback structure and describes the general feedback configuration. In section 5.2, the well-posedness of the feedback loop is defined. Next, the notion of internal stability is introduced and the relationship is established between the state space characterization of internal stability and the transfer matrix characterization of internal stability in section 5.3. The stable coprime factorizations of rational matrices are also introduced in section 5.4. Section 5.5 considers feedback properties and discusses how to achieve desired performance using feedback control. These discussions lead to a loop shaping control design technique which is introduced in section 5.6. Finally, we consider the mathematical formulations of optimal $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control problems in section 5.7.

5.1 Feedback Structure

In designing control systems, there are several fundamental issues that transcend the boundaries of specific applications. Although they may differ for each application and may have different levels of importance, these issues are generic in their relationship to control design objectives and procedures. Central to these issues is the requirement to provide satisfactory performance in the face of modeling errors, system variations, and uncertainty. Indeed, this requirement was the original motivation for the development of feedback systems. Feedback is only required when system performance cannot be achieved because of uncertainty in system characteristics. The more detailed treatment of model uncertainties and their representations will be discussed in Chapter 9.

For the moment, assuming we are given a model including a representation of uncertainty which we believe adequately captures the essential features of the plant, the next step in the controller design process is to determine what structure is necessary to achieve the desired performance. Prefiltering input signals (or open loop control) can change the dynamic response of the model set but cannot reduce the effect of uncertainty. If the uncertainty is too great to achieve the desired accuracy of response, then a feedback structure is required. The mere assumption of a feedback structure, however, does not guarantee a reduction of uncertainty, and there are many obstacles to achieving the uncertainty-reducing benefits of feedback. In particular, since for any reasonable model set representing a physical system uncertainty becomes large and the phase is completely unknown at sufficiently high frequencies, the loop gain must be small at those frequencies to avoid destabilizing the high frequency system dynamics. Even worse is that the feedback system actually increases uncertainty and sensitivity in the frequency ranges where uncertainty is significantly large. In other words, because of the type of sets required to reasonably model physical systems and because of the restriction that our controllers be causal, we cannot use feedback (or any other control structure) to cause our closed-loop model set to be a proper subset of the open-loop model set. Often, what can be achieved with intelligent use of feedback is a significant reduction of uncertainty for certain signals of importance with a small increase spread over other signals. Thus, the feedback design problem centers around the tradeoff involved in reducing the overall impact of uncertainty. This tradeoff also occurs, for example, when using feedback to reduce command/disturbance error while minimizing response degradation due to measurement noise. To be of practical value, a design technique must provide means for performing these tradeoffs. We will discuss these tradeoffs in more detail later in section 5.5 and in Chapter 6.

To focus our discussion, we will consider the standard feedback configuration shown in Figure 5.1. It consists of the interconnected plant P and controller K forced by

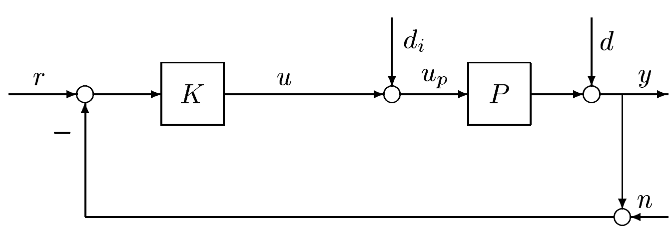

Figure 5.1: Standard Feedback Configuration

command r, sensor noise n, plant input disturbance d_i, and plant output disturbance d. In general, all signals are assumed to be multivariable, and all transfer matrices are assumed to have appropriate dimensions.

5.2 Well-Posedness of Feedback Loop

Assume that the plant P and the controller K in Figure 5.1 are fixed real rational proper transfer matrices. Then the first question one would ask is whether the feedback interconnection makes sense or is physically realizable. To be more specific, consider a simple example where

$$P = -\frac{s-1}{s+2}, \quad K = 1$$

are both proper transfer functions. However,

$$u = \frac{(s+2)}{3}(r - n - d) - \frac{s-1}{3}d_i$$

i.e., the transfer functions from the external signals $r - n - d$ and d_i to u are not proper. Hence, the feedback system is not physically realizable!

Definition 5.1 A feedback system is said to be *well-posed* if all closed-loop transfer matrices are well-defined and proper.

Now suppose that all the external signals r, n, d, and d_i are specified and that the closed-loop transfer matrices from them to u are respectively well-defined and proper. Then, y and all other signals are also well-defined and the related transfer matrices are proper. Furthermore, since the transfer matrices from d and n to u are the same and differ from the transfer matrix from r to u by only a sign, the system is well-posed if and only if the transfer matrix from $\begin{bmatrix} d_i \\ d \end{bmatrix}$ to u exists and is proper.

In order to be consistent with the notation used in the rest of the book, we shall denote

$$\hat{K} := -K$$

and regroup the external input signals into the feedback loop as w_1 and w_2 and regroup the input signals of the plant and the controller as e_1 and e_2. Then the feedback loop with the plant and the controller can be simply represented as in Figure 5.2 and the system is well-posed if and only if the transfer matrix from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to e_1 exists and is proper.

Lemma 5.1 *The feedback system in Figure 5.2 is well-posed if and only if*

$$I - \hat{K}(\infty)P(\infty) \tag{5.1}$$

is invertible.

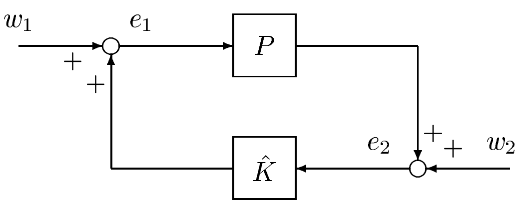

Figure 5.2: Internal Stability Analysis Diagram

Proof. The system in the above diagram can be represented in equation form as

$$\begin{aligned} e_1 &= w_1 + \hat{K} e_2 \\ e_2 &= w_2 + P e_1. \end{aligned}$$

Then an expression for e_1 can be obtained as

$$(I - \hat{K}P)e_1 = w_1 + \hat{K}w_2.$$

Thus well-posedness is equivalent to the condition that $(I - \hat{K}P)^{-1}$ exists and is proper. But this is equivalent to the condition that the constant term of the transfer function $I - \hat{K}P$ is invertible. □

It is straightforward to show that (5.1) is equivalent to either one of the following two conditions:

$$\begin{bmatrix} I & -\hat{K}(\infty) \\ -P(\infty) & I \end{bmatrix} \text{ is invertible;} \tag{5.2}$$

$$I - P(\infty)\hat{K}(\infty) \text{ is invertible.}$$

The well-posedness condition is simple to state in terms of state-space realizations. Introduce realizations of P and $\hat{K}$:

$$P = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \tag{5.3}$$

$$\hat{K} = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]. \tag{5.4}$$

Then $P(\infty) = D$ and $\hat{K}(\infty) = \hat{D}$. For example, well-posedness in (5.2) is equivalent to the condition that

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix} \text{ is invertible.} \tag{5.5}$$

Fortunately, in most practical cases we will have $D = 0$, and hence well-posedness for most practical control systems is guaranteed.

5.3 Internal Stability

Consider a system described by the standard block diagram in Figure 5.2 and assume the system is well-posed. Furthermore, assume that the realizations for $P(s)$ and $\hat{K}(s)$ given in equations (5.3) and (5.4) are *stabilizable and detectable.*

Let x and $\hat{x}$ denote the state vectors for P and $\hat{K}$, respectively, and write the state equations in Figure 5.2 with w_1 and w_2 set to zero:

$$\begin{aligned} \dot{x} &= Ax + Be_1 && (5.6)\\ e_2 &= Cx + De_1 && (5.7)\\ \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}e_2 && (5.8)\\ e_1 &= \hat{C}\hat{x} + \hat{D}e_2. && (5.9) \end{aligned}$$

Definition 5.2 The system of Figure 5.2 is said to be *internally stable* if the origin $(x, \hat{x}) = (0, 0)$ is asymptotically stable, i.e., the states $(x, \hat{x})$ go to zero from all initial states when $w_1 = 0$ and $w_2 = 0$.

Note that internal stability is a state space notion. To get a concrete characterization of internal stability, solve equations (5.7) and (5.9) for e_1 and e_2:

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}.$$

Note that the existence of the inverse is guaranteed by the well-posedness condition. Now substitute this into (5.6) and (5.8) to get

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}.$$

Thus internal stability is equivalent to the condition that $\tilde{A}$ has all its eigenvalues in the open left-half plane. In fact, this can be taken as a definition of internal stability.

Lemma 5.2 *The system of Figure 5.2 with given stabilizable and detectable realizations for P and $\hat{K}$ is internally stable if and only if $\tilde{A}$ is a Hurwitz matrix.*

It is routine to verify that the above definition of internal stability depends only on P and $\hat{K}$, not on specific realizations of them as long as the realizations of P and $\hat{K}$ are both stabilizable and detectable, i.e., no extra unstable modes are introduced by the realizations.

The above notion of internal stability is defined in terms of state-space realizations of P and $\hat{K}$. It is also important and useful to characterize internal stability from the transfer matrix point of view. Note that the feedback system in Figure 5.2 is described, in term of transfer matrices, by

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{5.10}$$

Now it is intuitively clear that if the system in Figure 5.2 is internally stable, then for all bounded inputs (w_1, w_2), the outputs (e_1, e_2) are also bounded. The following lemma shows that this idea leads to a transfer matrix characterization of internal stability.

Lemma 5.3 *The system in Figure 5.2 is internally stable if and only if the transfer matrix*

$$\begin{bmatrix} I & -\hat{K} \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I + \hat{K}(I - P\hat{K})^{-1}P & \hat{K}(I - P\hat{K})^{-1} \\ (I - P\hat{K})^{-1}P & (I - P\hat{K})^{-1} \end{bmatrix} \tag{5.11}$$

from (w_1, w_2) *to* (e_1, e_2) *belongs to* $\mathcal{RH}_\infty$.

Proof. As above let $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and $\left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right]$ be stabilizable and detectable realizations of P and $\hat{K}$, respectively. Let y_1 denote the output of P and y_2 the output of $\hat{K}$. Then the state-space equations for the system in Figure 5.2 are

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} &= \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} &= \begin{bmatrix} C & 0 \\ 0 & \hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{D} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \\ \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} &= \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. \end{aligned}$$

The last two equations can be rewritten as

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Now suppose that this system is internally stable. Then the well-posedness condition implies that $(I - D\hat{D}) = (I - P\hat{K})(\infty)$ is invertible. Hence, $(I - P\hat{K})$ is invertible. Furthermore, since the eigenvalues of

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}$$

are in the open left-half plane, it follows that the transfer matrix from (w_1, w_2) to (e_1, e_2) given in (5.11) is in $\mathcal{RH}_\infty$.

Conversely, suppose that $(I - P\hat{K})$ is invertible and the transfer matrix in (5.11) is in $\mathcal{RH}_\infty$. Then, in particular, $(I - P\hat{K})^{-1}$ is proper which implies that $(I - P\hat{K})(\infty) = (I - D\hat{D})$ is invertible. Therefore,

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}$$

is nonsingular. Now routine calculations give the transfer matrix from $\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ to $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ in terms of the state space realizations:

$$\begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1} \left\{ \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix} + \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} (sI - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix} \right\} \begin{bmatrix} I & -\hat{D} \\ -D & I \end{bmatrix}^{-1}.$$

Since the above transfer matrix belongs to $\mathcal{RH}_\infty$, it follows that

$$\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} (sI - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}$$

as a transfer matrix belongs to $\mathcal{RH}_\infty$. Finally, since (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are stabilizable and detectable,

$$\left(\tilde{A}, \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}, \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \right)$$

is stabilizable and detectable. It then follows that the eigenvalues of $\tilde{A}$ are in the open left-half plane. □

Note that to check internal stability, it is necessary (and sufficient) to test whether each of the four transfer matrices in (5.11) is in $\mathcal{RH}_\infty$. Stability cannot be concluded even if three of the four transfer matrices in (5.11) are in $\mathcal{RH}_\infty$. For example, let an interconnected system transfer function be given by

$$P = \frac{s-1}{s+1}, \qquad \hat{K} = -\frac{1}{s-1}.$$

Then it is easy to compute

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s+2} & -\dfrac{s+1}{(s-1)(s+2)} \\ \dfrac{s-1}{s+2} & \dfrac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix},$$

which shows that the system is not internally stable although three of the four transfer functions are stable. This can also be seen by calculating the closed-loop A-matrix with any stabilizable and detectable realizations of P and $\hat{K}$.

Remark 5.1 It should be noted that internal stability is a basic requirement for a practical feedback system. This is because all interconnected systems may be unavoidably subject to some nonzero initial conditions and some (possibly small) errors, and it cannot be tolerated in practice that such errors at some locations will lead to unbounded signals at some other locations in the closed-loop system. Internal stability guarantees that all signals in a system are bounded provided that the injected signals (at any locations) are bounded. ♡

However, there are some special cases under which determining system stability is simple.

Corollary 5.4 *Suppose $\hat{K} \in \mathcal{RH}_\infty$. Then the system in Figure 5.2 is internally stable iff $(I - P\hat{K})^{-1}P \in \mathcal{RH}_\infty$.*

Proof. The necessity is obvious. To prove the sufficiency, it is sufficient to show that $(I - P\hat{K})^{-1} \in \mathcal{RH}_\infty$. But this follows from

$$(I - P\hat{K})^{-1} = I + (I - P\hat{K})^{-1}P\hat{K}$$

and $(I - P\hat{K})^{-1}P, \hat{K} \in \mathcal{RH}_\infty$. □

This corollary is in fact the basis for the classical control theory where the stability is checked only for one closed-loop transfer function with the implicit assumption that the controller itself is stable. Also, we have

Corollary 5.5 *Suppose $P \in \mathcal{RH}_\infty$. Then the system in Figure 5.2 is internally stable iff $\hat{K}(I - P\hat{K})^{-1} \in \mathcal{RH}_\infty$.*

Corollary 5.6 *Suppose $P \in \mathcal{RH}_\infty$ and $\hat{K} \in \mathcal{RH}_\infty$. Then the system in Figure 5.2 is internally stable iff $(I - P\hat{K})^{-1} \in \mathcal{RH}_\infty$.*

To study the more general case, define

$$\begin{aligned} n_c &:= \text{number of open rhp poles of } \hat{K}(s) \\ n_p &:= \text{number of open rhp poles of } P(s). \end{aligned}$$

Theorem 5.7 *The system is internally stable if and only if*

(i) the number of open rhp poles of $P(s)\hat{K}(s) = n_c + n_p$;

(ii) $\phi(s) := \det(I - P(s)\hat{K}(s))$ has all its zeros in the open left-half plane (i.e., $(I - P(s)\hat{K}(s))^{-1}$ is stable).

Proof. It is easy to show that $P\hat{K}$ and $(I - P\hat{K})^{-1}$ have the following realizations:

$$P\hat{K} = \left[\begin{array}{cc|c} A & B\hat{C} & B\hat{D} \\ 0 & \hat{A} & \hat{B} \\ \hline C & D\hat{C} & D\hat{D} \end{array}\right]$$

$$(I - P\hat{K})^{-1} = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where

$$\begin{aligned} \bar{A} &= \begin{bmatrix} A & B\hat{C} \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B\hat{D} \\ \hat{B} \end{bmatrix} (I - D\hat{D})^{-1} \begin{bmatrix} C & D\hat{C} \end{bmatrix} \\ \bar{B} &= \begin{bmatrix} B\hat{D} \\ \hat{B} \end{bmatrix} (I - D\hat{D})^{-1} \\ \bar{C} &= (I - D\hat{D})^{-1} \begin{bmatrix} C & D\hat{C} \end{bmatrix} \\ \bar{D} &= (I - D\hat{D})^{-1}. \end{aligned}$$

It is also easy to see that $\bar{A} = \tilde{A}$. Hence, the system is internally stable iff $\bar{A}$ is stable.

Now suppose that the system is internally stable, then $(I - P\hat{K})^{-1} \in \mathcal{RH}_\infty$. This implies that all zeros of $\det(I - P(s)\hat{K}(s))$ must be in the left-half plane. So we only need to show that given condition (ii), condition (i) is necessary and sufficient for the internal stability. This follows by noting that $(\bar{A}, \bar{B})$ is stabilizable iff

$$\left(\begin{bmatrix} A & B\hat{C} \\ 0 & \hat{A} \end{bmatrix}, \begin{bmatrix} B\hat{D} \\ \hat{B} \end{bmatrix} \right) \tag{5.12}$$

is stabilizable; and $(\bar{C}, \bar{A})$ is detectable iff

$$\left(\begin{bmatrix} C & D\hat{C} \end{bmatrix}, \begin{bmatrix} A & B\hat{C} \\ 0 & \hat{A} \end{bmatrix} \right) \tag{5.13}$$

is detectable. But conditions (5.12) and (5.13) are equivalent to condition (i), i.e., $P\hat{K}$ has no unstable pole/zero cancelations. □

With this observation, the MIMO version of the Nyquist stability theorem is obvious.

Theorem 5.8 (Nyquist Stability Theorem) *The system is internally stable if and only if condition (i) in Theorem 5.7 is satisfied and the Nyquist plot of $\phi(j\omega)$ for $-\infty \le \omega \le \infty$ encircles the origin, $(0,0)$, $n_k + n_p$ times in the counter-clockwise direction.*

Proof. Note that by SISO Nyquist stability theorem, $\phi(s)$ has all zeros in the open left-half plane if and only if the Nyquist plot of $\phi(j\omega)$ for $-\infty \le \omega \le \infty$ encircles the origin, $(0,0)$, $n_k + n_p$ times in the counter-clockwise direction. □