Control System Design - LQG Part 2

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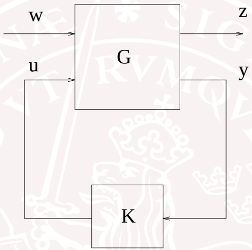
Lecture - LQG Design

- What do the “technical conditions” mean?
- Introducing integral action, etc
- Loop Transfer Recovery (LTR)
- Examples

For theory and more information, see PhD course on LQG

Reading tip: Ch 5 in Maciejowski, Multivariable Feedback Design
see home page for more links

Reminder - Notation



$$
\begin{aligned}
\dot{x} &= Ax + B_1 w + B_2 u \\
z &= C_1 x + D_{12} u \\
y &= C_2 x + D_{21} w + D_{22} u
\end{aligned}
$$

Reminder - Technical Conditions

1) $[A, B_2]$ stabilizable

2) $[C_2, A]$ detectable

3) "No zeros on imaginary axis" $u \to z$

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m \qquad \forall \omega$$

and D_{12} has full column rank (no free control)

4) "No zeros on imaginary axis" $w \to y$

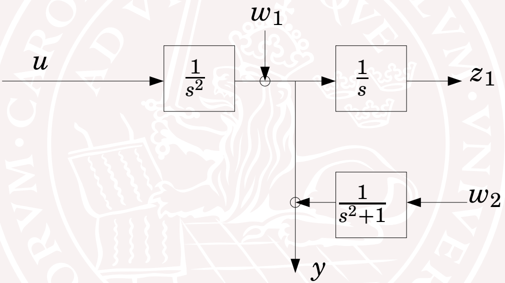
$$\text{rank} \begin{pmatrix} j\omega I - A & -B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p \qquad \forall \omega$$

and D_{21} has full row rank (no noise-free measurements)

Discrete time: change imag. axis $j\omega$ to unit circle $e^{j\omega}$.

An Example

Control of double integrator $G(s) = 1/s^2$ with frequency dependent weights (Question: Why would one choose such weights?)



TAT: The formulation violates the “technical conditions”, why?

The example is available as lqg2.m on home page

Answer

D_{12} not full rank

›Solution: New punished signal, $z_2 = \rho u$

Non-stabilizable, non-detectable modes

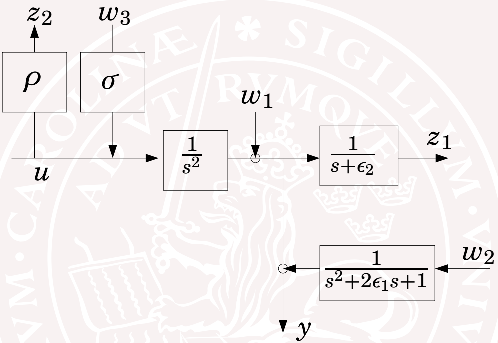
›Solution: Perturb $1/s$ and $1/(s^2+1)$ weights

The matrix $\begin{pmatrix} j\omega I - A & -B_1 \\ C_2 & D_{21} \end{pmatrix}$ looses rank in $\omega = 0$.

No input noise will lead to Kalman filter with $L_{opt} = 0$, which gives marginally unstable Kalman filter.

›Solution: Add input noise w_3 to process.

New System



$\rho = 0.1,\ \ \sigma = 0.01,\ \ \epsilon_1 = \epsilon_2 = 10^{-4}$ gives

$$C(s) = 12.85\frac{(s^2+0.1248s+0.00778)(s^2+0.0002s+1)}{(s+0.0001)(s^2+0.315s+1.768)(s^2+5.135s+11.96)}$$

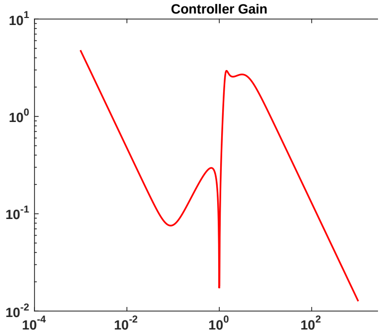
The Code

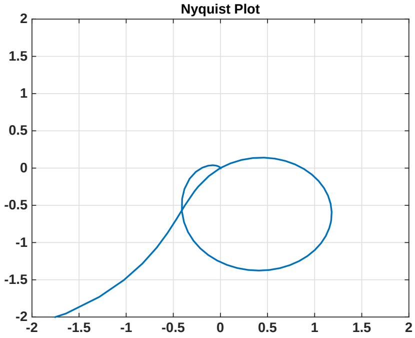
```
rho=0.1;ep1=0.0001;ep2=0.0001;sigma=0.01;
A=[0 1 0 0 0; 0 0 0 0 0 ; 0 0 -ep1 1 0;0 0 -1 -ep1 0 ; 1 0 0 0 -ep2];
B1=[0 0 0  ; 0 0 sigma ; 0 0 0 ; 0 1 0 ; 1 0 0];
B2=[0 ; 1 ; 0 ; 0 ; 0];
C1=[0 0 0 0 1; 0 0 0 0 0 ];
C2=[1 0 0 1 0];
D11=[0 0 0; 0 0 0 ];
D12=[0; rho];
D21=[1 0 0];
D22=0;

Q=C1'*C1;
R=D12'*D12;
N=C1'*D12;
[k,s,e] = lqr(A,B2,Q,R,N);
G=eye(length(A));
H=zeros(1,length(A));
syse = ss(A,[B2 G],C2,[D22 H])
R11 = B1*B1';
R22 = D21*D21';
R12 = B1*D21';
[kest,l,p]=kalman(syse,R11,R22,R12)

reg = zpk(lqgreg(kest,k));
frv=logspace(-3,3,1000);
fr1 = squeeze(freqresp(reg,frv));
loglog(frv,abs(fr1),'r','Linewidth',2)
```

Result

5th order controller with reasonable gain.

Integral action and notch at 1 rad/s.





Technical conditions - D_{21} not full rank

If D_{21} does not have full rank (i.e. $R_2 = D_{21}D_{21}^T$ not pos def), some measurements are error free.

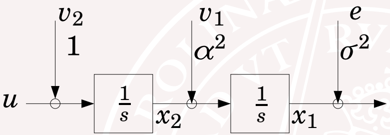
Can use y directly for calculation of some combination of states

Kalman filter gains will become very large, trying to make use of these error free directions. Resulting Kalman filter will be of lower order.

Luenberger observer - reduced observer of order n-rank(C)

See linear systems course

Example - Reduced order observer



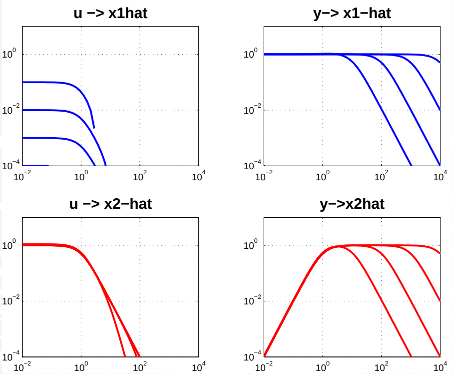
$$
\begin{aligned}
\dot{x}_1 &= x_2 + v_1 \\
\dot{x}_2 &= u + v_2 \\
y &= x_1 + e
\end{aligned}
$$

v_1, v_2, and e white noise, incr variance α^2, 1, and σ^2

Kalman gain for $\alpha = 1$ and $\sigma = 10^{-2},\ 10^{-4},\ 10^{-6},\ 10^{-8}$

$$
L = \begin{pmatrix} 10.95 \\ 10 \end{pmatrix}, \quad L = \begin{pmatrix} 101 \\ 100 \end{pmatrix}, \quad L = \begin{pmatrix} 1001 \\ 1000 \end{pmatrix}, \quad L = \begin{pmatrix} 10001 \\ 10000 \end{pmatrix}
$$

Observer $\hat{x} = (sI - A + LC)^{-1}(Bu + Ly)$

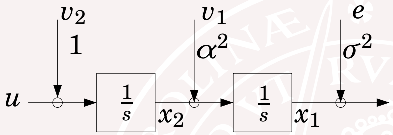


In the limit, $y = x_1$ is noise-free and can be derivated, giving

$$\begin{aligned} \dot{x}_2 &= u + v_2 \\ y_{new} &= \dot{y} = x_2 + v_1 \end{aligned}$$

Kalman filter for the new system $s\hat{x}_2 = u + L_{new}(y_{new} - \hat{x}_2)$

Example - Reduced order observer



$$
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\end{aligned}
$$

v_1, v_2, and e white noise, incr variance α^2, 1, and σ^2

Optimal filter as $\sigma \to 0$ is first order:

$$
\begin{aligned}
\hat{x}_1 &= y \\
\hat{x}_2 &= \frac{\alpha}{\alpha s + 1} u + \frac{s}{\alpha s + 1} y
\end{aligned}
$$

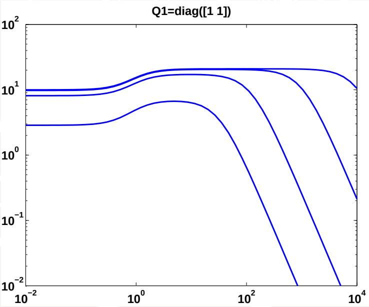
$\hat{x}_2 \approx \frac{1}{s} u$ if α large

$\hat{x}_2 \approx s y$ if α small

Technical conditions D_{12} not full rank

If $Q_2 = D_{12}^T D_{12}$ is not pos. def then some combinations of control signals are free. States can be moved freely and infinitly fast in some directions.

For the system above, the LQG controller obtained with $Q_2 = 10^{-2}, \ldots, \; 10^{-8}$ tends to a lead-filter ($\alpha = 1, \; \sigma = 0$)



Technical Conditions - More Intuition

The following system

$$\begin{aligned} \dot{x} &= 0 \\ y &= x + e \end{aligned}$$

fails condition 4:

$$\begin{pmatrix} sI - A & -B_1 \\ C_2 & D_{21} \end{pmatrix} = \begin{pmatrix} s & 0 \\ 1 & 1 \end{pmatrix} \quad \text{looses rank for } s = 0$$

what happens with the observer in stationarity?

Technical Conditions - More Intuition

Optimal observer is

$$\hat{x} = \frac{1}{t}\int_0^t y\,dt$$

$$\frac{d\hat{x}}{dt} = \frac{1}{t}(y - \hat{x})$$

so Kalman filter gain is time varying with

$$L(t) = \frac{1}{t} \to 0$$

and the observer system becomes marginally stable:

$$A - L(t)C_2 \to 0$$

Lecture - LQG Design

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- Introducing integral action, etc
- Loop Transfer Recovery (LTR)
- Examples

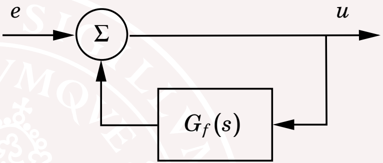
Disturbance Modeling

Integral action and generalized integral control can be generated by disturbance modeling.

1. Recall integral action and generalized integral action
2. Choose controller that gives integral action
3. Models disturbances and incorporate the models in the controller
4. Disturbances can be load disturbances, measurement noise, reference values.

Generalized Integral Control

- Constant but unknown
- Ramps with unknown levels and rates
- Sinusoidal with known frequency but unknown amplitude
- Periodic with known period but unknown shape



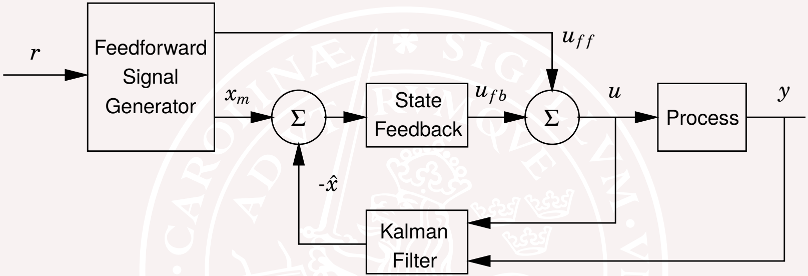
$$C(s) = \frac{k}{1 - G_f(s)}$$

$$G_f(s) = \frac{1}{1 + sT} \qquad C_{const}(s) = 1 + \frac{1}{sT}$$

$$G_f(s) = \frac{2\zeta\omega_0 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \qquad C_{sine}(s) = \frac{s^2 + 2\zeta\omega_0 s + \omega_0^2}{s^2 + \omega_0^2}$$

$$G_f(s) = e^{-sL} \qquad C_{periodic}(s) = \frac{1}{1 - e^{-sL}}$$

A General 2DOF System



The signals x_m and $y_m = Cx_m$ give the desired responses of states and output, the signal u_{ff} drives the system in the desired way

Many ways to generate command signals

- Tables, dynamic models
- Constraints can be accounted for
- Feedforward design requires inversion of process dynamics, see BottomUp lecture

Model Following

Desired behavior

$$\frac{dx_m}{dt} = Ax_m + Bu, \qquad y_m = Cx_m.$$

Desired feedforward signal $u = u_{ff}$ can be generated in many ways. See lecture on Bottom Up.

Integral Action by Disturbance Observer

Process and unknown but constant load disturbance v

$$\frac{dx}{dt} = Ax + B(u + v), \qquad y = Cx, \qquad \frac{dv}{dt} = 0.$$

Augment process state x by disturbance state v gives the following model for the *the process and its disturbances*

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u, \qquad y = \begin{pmatrix} C & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix}$$

Is the state v stabilizable? Does it matter? Observer

$$\frac{d}{dt}\begin{pmatrix} \hat{x} \\ \hat{v} \end{pmatrix} = \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{v} \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} L_x \\ L_v \end{pmatrix}(y - C\hat{x}),$$

Controller

$$u = u_{ff} + K_x(x_m - \hat{x}) - \hat{v}$$

where u_{ff} is the feedforward signal and x_m is the desired state from the feedforward signal generator.

The Complete Controller

Introducing $\tilde{x} = x_m - \hat{x}$ and eliminate $\hat{x}$

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= (A - BK_x - L_x C)\tilde{x} + L_x(y_m - y) = A_c\tilde{x} + L_x(y_m - y) \\
\frac{d\hat{v}}{dt} &= L_v C\tilde{x} - L_v(y_m - y) \\
u &= u_{ff} + K_x\tilde{x} - \hat{v}
\end{aligned}$$

Replace $\hat{v}$ by $w = L_v C A_c^{-1}\tilde{x} - \hat{v}$. Then the controller becomes

$$\begin{aligned}
\frac{d\tilde{x}}{dt} &= A_c\tilde{x} + L_x(y_m - y) \\
\frac{dw}{dt} &= (L_v + L_v C A_c^{-1} L_x)(y_m - y) = k_i(y_m - y) \\
u &= u_{ff} + (K_x - L_v C A_c^{-1})\tilde{x} + w.
\end{aligned}$$

Controller Transfer Function

$$\begin{aligned}\frac{d\tilde{x}}{dt} &= A_c\tilde{x} + L_x(y_m - y)\\ \frac{dw}{dt} &= (L_v + L_v C A_c^{-1} L_x)(y_m - y) = K_i(y_m - y)\\ u &= u_{ff} + (K_x - L_v C A_c^{-1})\tilde{x} + w.\end{aligned}$$

Transfer function is

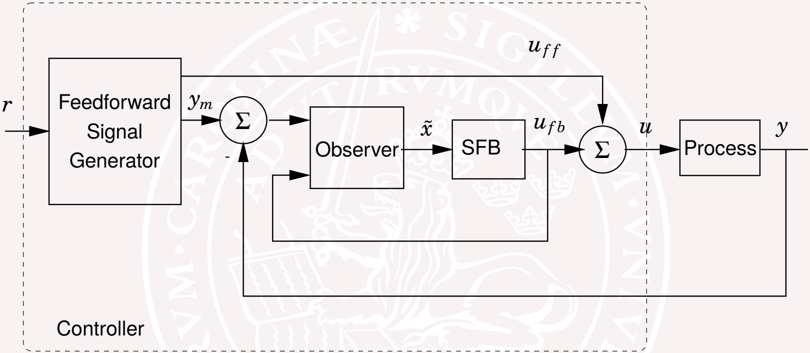
$$C(s) = G_{u_{fb}e} = \frac{L_v}{s} + \Big(K_x - \frac{1}{s}L_v C\Big)(sI - A + BK_x + L_x C)^{-1} L_x$$

Integral gain

$$k_i = L_v(1 + CA_c^{-1}L_x) = L_v\Big(1 + C(A - BK_x - L_x C)^{-1} L_x\Big).$$

Since the integrator is separate it is easy to deal with manual control and anti-windup.

Block Diagram



- Notice that feedforward generator only delivers y_m and u_{ff}
- Practical consequences which is useful if you would like to change state representations (Layering)

Other Disturbances

Sinusoidal disturbances with frequency ω_d can be captured by the model

$$\frac{dv}{dt} = \begin{pmatrix} 0 & \omega_d \\ -\omega_d & 0 \end{pmatrix} v$$

We can deal with any disturbance that can be generated by

$$\frac{dw}{dt} = A_v w, \qquad v = C_v w$$

The matrix A_v often has eigenvalues on the imaginary axis. Why?

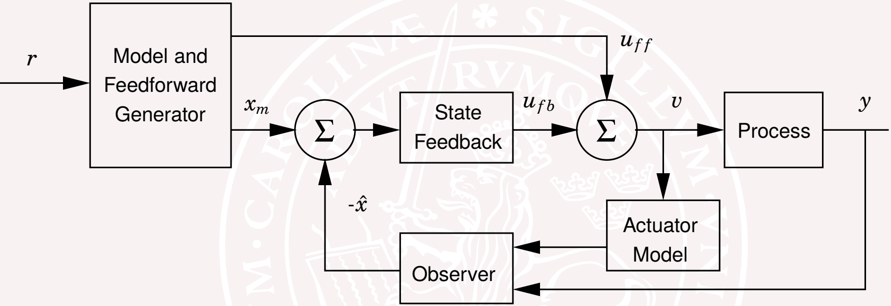
Controller Dynamics vs Disturbance Dynamics

$$\frac{dw}{dt} = A_v w, \qquad v = C_v w$$

If w is added to the input noise to the process, the resulting LQG controller will typically have **poles at the eigenvalues of A_v**

If w is added as measurement noise, the resulting LQG controller will typically have **transmission zeros at the eigenvalues of A_v**

Avoiding Windup in State Based Controller



- Dont fool the observer!
- Model predictive control (optimization with constraints)
- Easy to obtain tracking mode

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + L(y - C\hat{x}), \quad u = \mathrm{sat}(v), \quad v = u_{ff} + K(x_m - \hat{x})$$

Reference Values

Process model and controller

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx, \qquad u = -Kx + K_r r.$$

Closed loop system

$$\frac{dx}{dt} = (A - BK)x + BK_r r, \qquad y = Cx.$$

Steady state output

$$x_0 = (A - BK)^{-1} BK_r r.$$

Inverse exist because $A - BK$ is stable. Choosing

$$K_r = \left(C(A - BK)^{-1} B\right)^{-1}$$

gives the output $y = r$. This choice gives a *calibrated system*. The correct steady state is maintained by carefully matching the feedforward gain K_r to the system parameters.

Explicit Integral Action

Process model

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

Controller (forced integral action)

$$u = -K(x - x_m) - k_i z, \qquad \frac{dz}{dt} = Cx - r$$

Augment process state by the integrator state z

$$\frac{d}{dt}\begin{pmatrix} x \\ z \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} u \\ r \end{pmatrix}, \qquad y = \begin{pmatrix} C & 0 \end{pmatrix}$$

- Many design methods, pole-placemet, LQG, etc
- Condition for reachability and observability?
- When will it work

Reference Signals and Integrator in LQG

Extend system with integrators (on the tracked outputs)

$$\dot{x}_i = r - y$$

$$\min \int x^T Q_1 x + u^T Q_2 u + x_i^T Q_3 x_i$$

gives $\begin{pmatrix} K_x & K_i \end{pmatrix}$.

Extended system is controllable, but x_i is noise-free so nonstandard Kalman filter (D_{21} not full rank). Reduced order observer.

Kalman filter L obtained from original system; don't estimate x_i

Reference Signals and Integrator in LQG

Use controller

$$u = -K_x\hat{x} - K_i x_i$$

or if feedforward signal u_{ff} is available

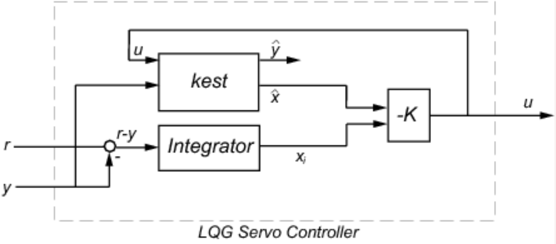
$$u = -K_x\hat{x} - K_i x_i + u_{ff}$$

Increased model order

Observer order not increased

Reference Signals in LQG

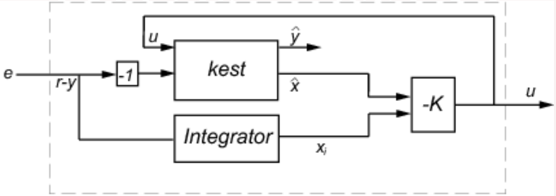
If r and y available separatly (2-DOF) one can do as follows (assuming dimensions of r and y are equal)



$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} A - BK_x - LC + LDK_x & -BK_i + LDK_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ x_i \end{pmatrix} + \begin{pmatrix} -L & L \\ I & -I \end{pmatrix} \begin{pmatrix} r \\ y \end{pmatrix}$$

Reference Signals in LQG

If only tracking error $e = r - y$ is available (1-DOF)



LQG Servo Controller

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} A - BK_x - LC + LDK_x & -BK_i + LDK_i \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \hat{x} \\ x_i \end{pmatrix} + \begin{pmatrix} -L \\ I \end{pmatrix} (r - y)$$

Reference Signals in LQG

If we know a stochastic model of the reference signal we can use this in the optimization.

Reference signal generated by linear system driven by white noise.

$$\begin{aligned}\dot{x}_r &= A_r x_r + B_r w \\ r &= C_r x_r + D_r w\end{aligned}$$

Augment the system with this model and use the general LQG framework described on previous lecture. Treat w as one of the disturbances and r as a known signal.

Reference Signals in LQG

If we have knowledge of future reference signals, r, this can be used to improve tracking performance further.

Introduce a Model and Feedforward Generator as above

$$u = K(x_m - \hat{x}) + u_{ff}$$

where u_{ff} is an open loop control signal that ideally produces the desired time variation x_m in process states.

Here u_{ff} and x_m can be non-causal functions of the reference signal r if this is known in advance

Lecture - LQG Design

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- Introducing integral action, etc
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Spectral factorisation - revisited

Assume $R_{12} = 0$

$$\begin{aligned} \dot{x} &= Ax + v, \qquad E(v_t v_{t-\tau}^T) = R_1 \delta_\tau \\ y &= C_2 x + e, \qquad E(e_t e_{t-\tau}^T) = R_2 \delta_\tau \end{aligned}$$

$$0 = AP + PA^T + R_1 - PC_2^T R_2^{-1} C_2 P, \;\; L = PC_2^T R_2^{-1}$$

"Equivalent" representation of y

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + L\epsilon \\ y &= C_2 \hat{x} + \epsilon \end{aligned}$$

where ϵ is the “innovation” process. Can show $E(\epsilon_t \epsilon_{t-\tau}^T) = R_2 \delta_\tau$

We can now write the spectrum of y in two different ways

Spectral factorisation - revisited

$$\begin{aligned}\Phi_y &= \Phi_e + C_2(sI - A)^{-1}\Phi_v(-sI - A^T)^{-1}C_2^T \\ \Phi_y &= [I_p + C_2(sI - A)^{-1}L]\Phi_\epsilon[\ldots]^*\end{aligned}$$

Hence

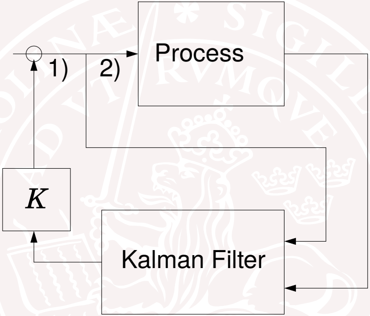
$$\begin{aligned}&R_2 + C_2(sI - A)^{-1}R_1(-sI - A^T)^{-1}C_2^T \\ &\quad = [I_p + C_2(sI - A)^{-1}L]R_2[I_p + C_2(-sI - A^T)^{-1}L]^T\end{aligned}$$

Kalman filter identity.

Compare previous lecture for the dual result (RDF)

If $R_{12} = 0$ then Kalman loop gain $C_2(sI - A)^{-1}L$ has same nice robustness as $K(sI - A)^{-1}B_2$ has when $Q_{12} = 0$

Influence of an observer



Loop gain at 1) is $G_1 = K(sI - A)^{-1}B_2$ but at 2)
$G_2 = K(sI - A + B_2K + LC_2)^{-1}LC_2(sI - A)^{-1}B_2$ (if $D_{22} = 0$)
Examples show one may loose *all robustness*

What happens if $L \to \infty$ (fast Kalman filter)?

LQG/LTR 1

Loop Transfer Recovery

Want to make G_2 as robust as G_1

References:

- Doyle and Stein, AC79, p. 607-611
- Doyle and Stein, AC81, p. 4-16

First LTR-method: Use fast (in a special way) observer

Sacrifice "noise optimality"

Almost like using an inverse for reconstruction

Not applicable if RHPL Zeros

LQG/LTR 1

First LTR-method: Add fictious input noise :

$$R_1 := R_1 + qB_2B_2^T$$

For square, minimum phase systems this gives $L \to \infty$ and

$$\lim_{q\to\infty} G_{LQG}(s)G(s) = K(sI - A)^{-1}B_2$$

Easy to try this idea, doesn't always lead to good designs.

Usually improves the robustness margins.

Dont let q go all the way to ∞.

Same problem as with all designs with fast observers

LQG/LTR 2

Second LTR-method: Punish more in output direction

$$Q_1 := Q_1 + qC_2^T C_2,$$

(ie use "cheap control")

Makes loop gain approach

$$\lim_{q \to \infty} G(s) G_{LQG}(s) = C_2 (sI - A)^{-1} L$$

ie the Kalman filter loop gain

Same problem as with all "cheap control" designs

LTR1 polynomial interpretation, SISO

System

$$C_2(sI - A)^{-1}B_2 = \frac{B(s)}{A(s)}$$

Disturbance influence

$$C(sI - A)^{-1}B_v = \frac{B_v(s)}{A(s)}$$

and $R_1 = B_v B_v^T, \qquad R_2 = 1$

Kalman filter identity

$$\begin{aligned} 1 \quad &+ \quad C(sI - A)^{-1}R_1(-sI - A^T)^{-1}C^T \\ &= \quad \left[1 + C(sI - A)^{-1}L\right]\left[1 + C(-sI - A)^{-1}L\right]^T \end{aligned}$$

or

$$\begin{aligned} A(s)A(-s) + B_v(s)B_v(-s) \quad &= \quad [A(s) + L(s)]\,[A(-s) + L(-s)] \\ &= \quad A_o(s)A_o(-s) \end{aligned}$$

LTR1 polynomial interpration, SISO

LTR-modification: $R_1^{mod} = R_1 + q^2 BB^T$ gives

$$\begin{aligned}
&C(sI - A)^{-1} R_1^{mod} (-sI - A^T)^{-1} C^T \\
&\quad = \frac{B_v(s)B_v(-s) + B(s)q^2 B(-s)}{A(s)A(-s)}
\end{aligned}$$

so

$$\begin{aligned}
A(s)A(-s) \quad &+ \quad B_v(s)B_v(-s) + B(s)q^2 B(-s) \\
&= \quad \Big[A(s) + L^{mod}(s)\Big]\Big[A(-s) + L^{mod}(-s)\Big] \\
&= \quad A_o^{mod}(s) A_o^{mod}(-s)
\end{aligned}$$

LTR1 polynomial interpration, SISO

Now for very large q

$$A_o^{mod}(s)A_o^{mod}(-s) \approx (-s^2)^n + B(s)q^2B(-s)$$

gives (according to root-locus discussion) if $B(s)$ stable

$$A_o^{mod}(s) \approx B(s)A_k(s), \quad A_k(s)A_k(-s) = b_0^{-2}((-s^2)^k + q^2)$$

where $k = \text{deg}A(s) - \text{deg}B(s)$.

LTR1 polynomial interpration, SISO

If we write the LQG controller as $U = -\frac{S(s)}{R(s)}Y$ we want to show that looptransfer in LQG

$$C(sI - A)^{-1}BK(sI - A + BK + LC)^{-1}L =: \frac{B(s)}{A(s)}\frac{S(s)}{R(s)}.$$

approaches the loop transfer in LQ

$$K(sI - A)^{-1}B =: \frac{K(s)}{A(s)}$$

LTR1 polynomial interpration, SISO

Closed loop polynomial is

$$A_c(s)A_o^{mod}(s) = A(s)R(s) + B(s)S(s)$$

and after some thought (for fixed s as $q \to \infty$)

$$R(s) \approx B(s)A_k(s), \quad S(s) \approx \frac{q}{b_0}\left[A_c(s) - A(s)\right] = \frac{q}{b_0}K(s)$$

so the loop transfer is now

$$\frac{B(s)}{A(s)}\frac{S(s)}{R(s)} \approx \frac{B(s)}{A(s)} \underbrace{\frac{q}{b_0 A_k(s)}}_{\approx 1} \frac{K(s)}{B(s)} \approx \frac{K(s)}{A(s)}$$

and we have the nice LQR-robustness over most frequencies

Lecture - LQG Design

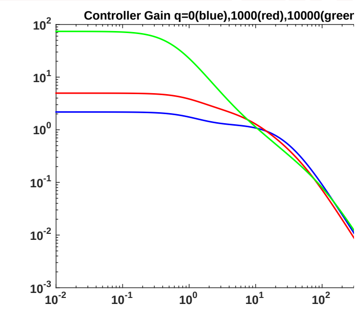
- What do the “technical conditions” mean?
- Introducing integral action, etc
- Loop Transfer Recovery (LTR)
- Examples

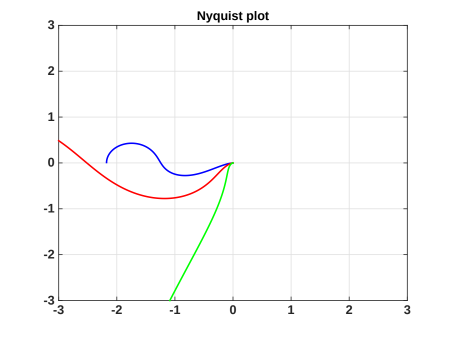
LTR Example Doyle-Stein, AC-79

$$G(s) = \frac{s+2}{(s+1)(s+3)}$$

```
A  = [-4 -3 ; 1 0];
B  = [1;0];
Bv = [-61;35];
C  = [1 2];
D  = 0;
Q1 = 80*[1 sqrt(35)]'*[1 sqrt(35)];
Q2 = 1;
R1 = Bv*Bv'+q*B*B';
R2 = 1;
```

Doyle-Stein, AC-79 Results





Better robustness obtained, with low extra cost (red)

Code available at home page: lqg3.m

Aircraft Design Example

Vertical-plane aircraft dynamics, from Maciejowski Ch 5.8

Inputs

- Spoiler angle (tenths of degree)
- Forward acceleration (m/s^2)
- Elevator angle (tenths of degree)

States

- Altitude (m)
- Forward speed (m/s)
- Pitch angle (degrees)
- Pitch rate (deg/s)
- Vertical speed (m/s)

Aircraft Design Example

```
A=[0         0    1.1320         0   -1.0000;
   0   -0.0538   -0.1712         0    0.0705;
   0         0         0    1.0000         0;
   0    0.0485         0   -0.8556   -1.0130;
   0   -0.2909         0    1.0532   -0.6859];

B=[      0         0         0;
   -0.1200    1.0000         0;
         0         0         0;
    4.4190         0   -1.6650;
    1.5750         0   -0.0732];

C=[1     0     0     0     0;
   0     1     0     0     0;
   0     0     1     0     0];
```

Aircraft LTR Design

Wanted:

- bandwidth of 10 rad/s: $\underline{\sigma}(T(i10)) = -3$dB
- integral action and reference tracking
- well-damped responses

Will use LTR2, gives loop gain approaching $C_2(sI - A)^{-1}L$

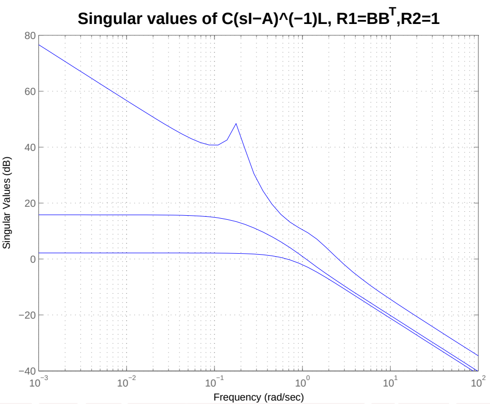
Aircraft LTR Design

Design performed in the following steps

1. Start to design Kalman filter, Guess: $R_1 = B_2 B_2^T$, $R_2 = 1$
2. Introduce integrators $w = \frac{1}{s+\epsilon}\nu$
3. ν colored noise: $R_1 = B_2(I + 9xx^T)B_2$ with clever x
4. Increase bandwidth, $R_1 := 100R_1$
5. Trim $S(i\omega)$ at 5.5 rad/s
6. LTR2, cheap control, $\rho = 10^{-6}$

Code available at home page: mac58.m

Gain of $C_2(sI - A)^{-1}L$, $R_1 = B_2B_2^T$, $R_2 = 1$



Need to introduce integral action

Extend system with integrators at input

Extended system

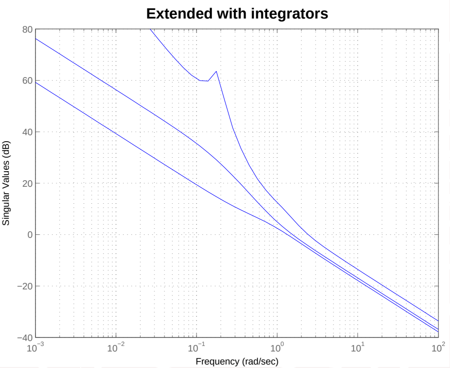
Introduce an integrator model for the input noise

$$w = \frac{1}{s + 10^{-4}} I_3 \nu$$

```
Aw=-0.0001*eye(3);Bw=eye(3);Cw=eye(3);Dw=zeros(3);
Aa=[    A       B1*Cw;
    zeros(3,5)    Aw];
Ba=[   B;  zeros(3)];
Ca=[C zeros(3)];
Da=zeros(3);
B1a=[B1*Dw;  Bw    ];
```

Principal Gains of $C_a(sI - A_a)^{-1}L_a$

Kalman filter loop gain for extended system



Would like to increse the lower singular value for $\omega = 10^{-3}$

Shaping Gains of $C_a(sI - A_a)^{-1}L_a$

Spectral factorisation identity if $R_2 = I$

$$[I + C_a(sI - A_a)^{-1}L_a][\dots]^* = G(s)R_1G^T(-s) + I$$

Select a frequency ω_0 and compute the SVD

$$G(j\omega_0)R_1^{1/2} = U\Sigma V^* = \sum_{i=1}^{m} \sigma_i u_i v_i^*$$

Now changing

$$R_1^{1/2} := R_1^{1/2}(I + \alpha v_j v_j^*)$$

gives

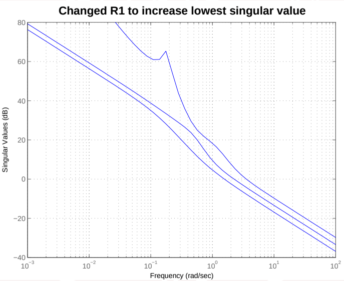
$$G(j\omega_0)R_1^{1/2}(I + \alpha v_j v_j^*) = \sum_{i \neq j}^{m} \sigma_i u_i v_i^* + (1 + \alpha)\sigma_j u_j v_j^*$$

One specific singular value of $I + C_a(sI - A_a)^{-1}L_a$ has been moved

Trimming R_1

Put $R_1 := R_1 + 9xx^T$ where x is the smallest singular vector direction for $C_a(j0.001I - A_a)^{-1}B_{1a}$ (think)

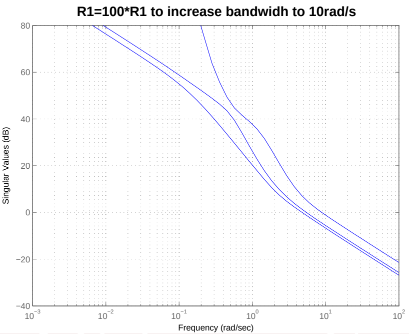
```
s=0.001*i;
Gf=Ca*inv(s*eye(size(Aa))-Aa)*B1a;
[u3,s3,v3]=svd(Gf);
v3real=real(v3(:,3));           % real approximation
R1sqrt=eye(3)+9*v3real*v3real'; % alfa=9, cause we want to change the gain
R1=R1sqrt*R1sqrt';              % of sigma_smallest with a factor 10
```



Need to increase bandwidth to 10rad/s

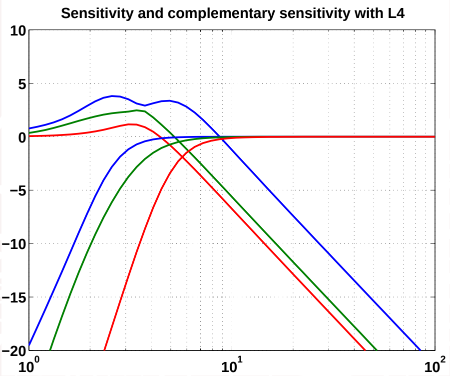
Further trimming R_1

Find factor by trial and error, gives $R_1 := 100R_1$



Lets have a look on the output sensitivity and complementary sensitivity
$S = [I_p + C_a(sI - A_a)^{-1}L_4]^{-1}$ and $T = I_p - S$

S and T



3-dB bandwidth for S around 2.5-5.5rad/s

3-dB bandwidth for T around 6.5-12 rad/s

Lets put all singular values of $S = -3$dB at 5.25rad/s

Improved S

Do a singular value decomposition of

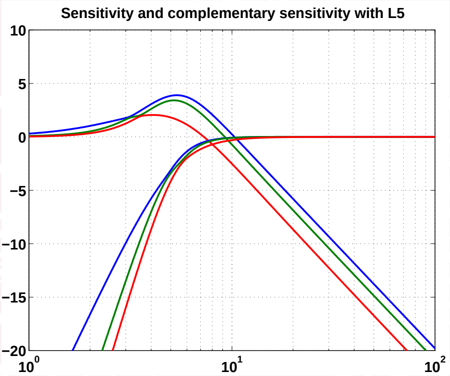
$$C_a(j5.25I - A)^{-1}\tilde{B}_1 = U\Sigma V^*$$

where $\tilde{B}_1\tilde{B}_1' = R_1$ with the new R_1

Change R_1 to get equal gain at 5.25 rad/s

```
Gf5=Ca*inv(5.225*i*eye(size(Aa))-Aa)*B1a*R1sqrt;
[u5,s5,v5]=svd(Gf5);
% Principal gains of the return difference with the current de
Ff=svd(eye(3)+Ca*inv(5.225*i*eye(size(Aa))-Aa)*L4);
% We want these to be sqrt(2)*[1 1 1]'.
% Principal gains of Gf is given by
Gfsv=sqrt(Ff.^2-1);
% and Ff=sqrt(2)*[1 1 1]' is equivalent to Gf=[1 1 1]', so
alfa=[1 1 1]'./Gfsv-ones(3,1);
R1sqrt=10*R1sqrt*(eye(3)+alfa(1)*real(v5(:,1))*real(v5(:,1))')
      (eye(3)+alfa(2)*real(v5(:,2))*real(v5(:,2))')*...
```

Result

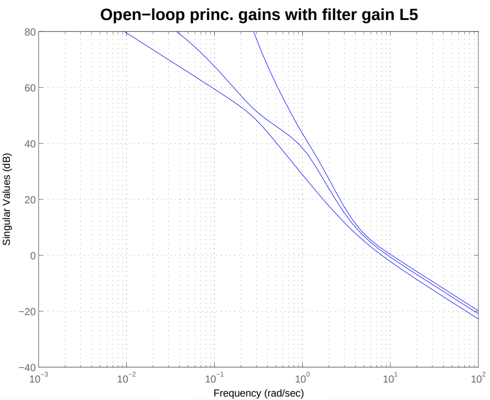


S(5.25) $\sim$ -3dB in all three directions, $\underline{\sigma}(T(10)) \sim -3$dB.

Let's use this Kalman filter gain L_5 !

Loop Gain $L(s) = C_a(sI - A)^{-1}L_5$

```
sigma(ss(Aa,L5,Ca,Da),wv);
```

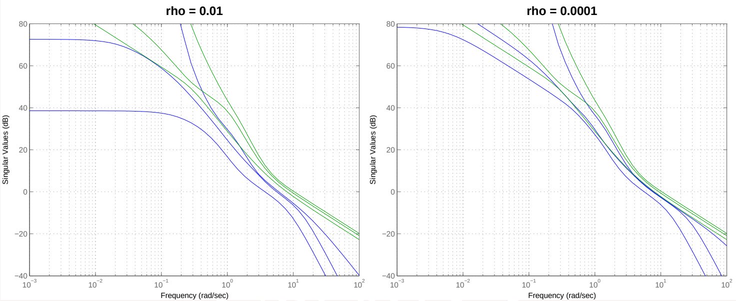


LTR recovery step $\rho = 10^{-2},\ \ 10^{-4}$

Use $Q_1 = C_a^T C_a$, $Q_2 = \rho I$ (and $Q_{12} = 0$)

Try $\rho = 10^{-2},\ \ 10^{-4}$

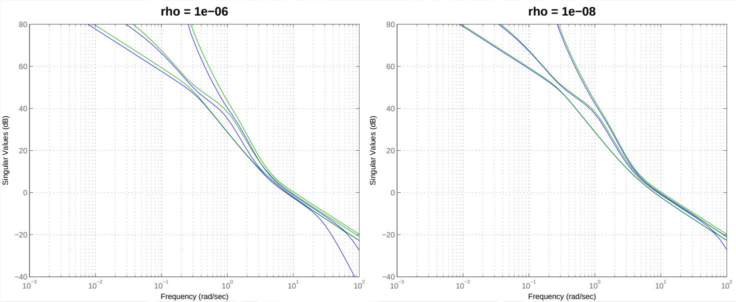
GG_{LQG} (blue) vs $K(sI - A)^{-1}B$ (green)



Need more LTR

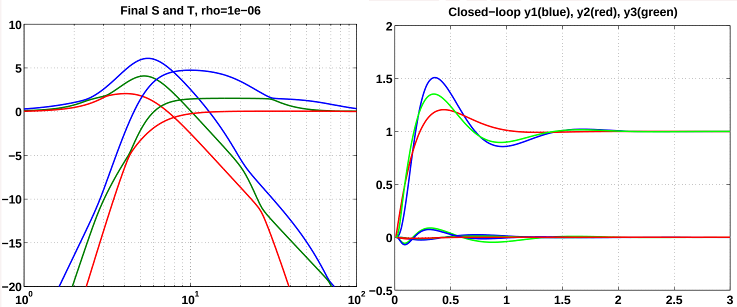
LTR recovery step $\rho = \rho = 10^{-6},\ \ 10^{-8}$

Try $\rho = 10^{-6},\ \ 10^{-8}$



$\rho = 10^{-6}$ seems ok (Maciejowski prefers $\rho = 10^{-8}$)

Result with $\rho = 10^{-6}$



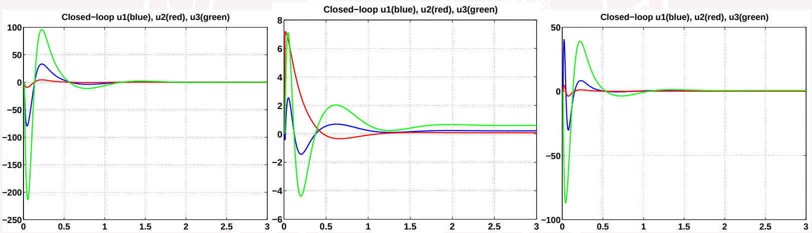
Quite good decoupling

$\overline{\sigma}(T) \sim 2$

Result with $\rho = 10^{-6}$

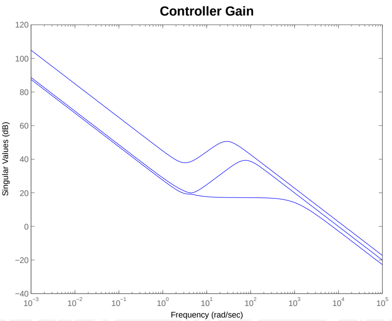
Steps in

- altitude 1 meter (left)
- forward speed 1 m/s (middle)
- pitch angle 1 degree (right)



Quite large control signals (20 deg elevator angle)

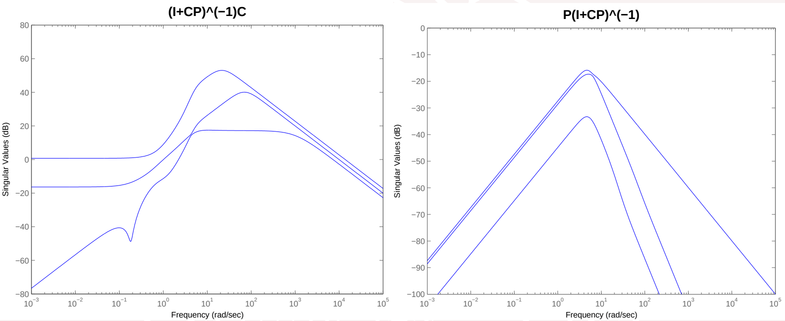
Controller Gain, with $\rho = 10^{-6}$



Quite high controller gains

$$K=\begin{bmatrix} -598.04 & -108.80 & 764.46 & 11.40 & 25.68 & 1 & 0 & 0 \\ -66.75 & 994.00 & 82.07 & 1.286 & 3.04 & 0 & 1 & 0 \\ -798.67 & -2.364 & -666.35 & -24.90 & 56.43 & 0 & 0 & 1 \end{bmatrix}$$

Gang of Four, $(I+CP)^{-1}C$ and $P(I+CP)^{-1}$



To really evaluate if this is a satisfactory design requires more domain knowledge

Hopefully a good initial design

Summary

LQG is a useful design method that extends well to MIMO

Can handle joint minimization of several criteria, e.g. the GangOfFour, and use model knowledge of disturbances

It can be hard to find weighting matrices achieving what you want

Extending the system, e.g. with integrators or other dynamic weights might be needed

Loop Transfer Recovery can be helpful to improve robustness, but dont overdo it