Robust Control, H_∞, ν and Glover-McFarlane

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Robust Control

1. MIMO performance
2. Robustness and the H_∞-norm
3. H_∞-control
4. ν-gap metric
5. Glover-MacFarlane

Theme: You get what you ask for!

References

Rantzer, PhD Course in Robust Control, 2015

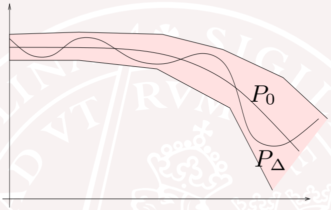
Zhou, Doyle, Glover, Robust Optimal Control 1996

Zhou, Doyle, Essentials of Robust Control, 1998

Doyle, Glover, Khargonekar, Francis, State Space Solutions to standard H_2 and H_∞ control problems, IEEE TAC 1989

Matlab: `hinfsyn`, `mixsyn`, `loopsyn`, `gapmetric`, `DynamicSystem/norm`, `dksyn`

Robust Control - Introduction



Why use feedback ?

Keep error small in spite of

- unknown disturbances
- model vs process mismatch

Can model/plant mismatch be taken care of by adding ficticious disturbances?

No, disturbances can e.g. not make system unstable

Robust stability vs robust performance

(Output) sensitivity function

$$S := (I + PC)^{-1}$$

- Nominal stability(NS): S stable
- Nominal performance(NP): $\overline{\sigma}(S) \leq 1/|W_p|$, where $W_p(s)$ weight
- Robust stability(RS): $S_\delta := (I + P_\delta C)^{-1}$ stable, $\forall P_\delta \in \mathcal{P}$
- Robust performance(RP): $\overline{\sigma}(S_\delta) \leq 1/|W_p|$, $\forall P_\delta \in \mathcal{P}$

For SISO systems NP + RS $\Rightarrow$ RP (more or less, will show later)

For MIMO systems NP + RS $\not\Rightarrow$ RP

Multiviariable effects make simple analysis dangerous

Motivating example [Skogestad]

Plant model [Distillation column]

$$P(s) = \frac{1}{50s+1}\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix}$$

Choose $C(s) = \frac{1}{s}P(s)^{-1}$ (dynamic decoupling)

Loop gain $PC = \frac{1}{s}I$

Closed-loop : $T(s) = PC(I+PC)^{-1} = \frac{1}{s+1}I$

Nice decoupled first order responses with time constant 1.

Example -continued

In reality: 20 percent input uncertainty (e.g. valve variations)

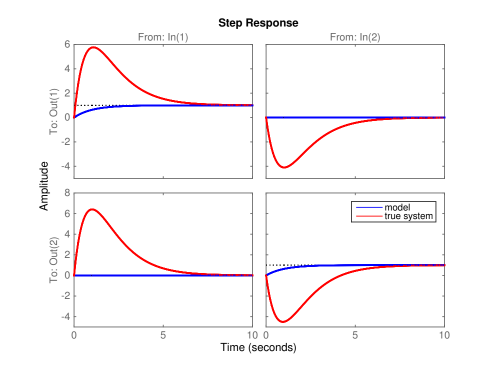
True control signal is $u_{i,p} = u_i(1+\delta_i)$ with $|\delta_i| < 0.2$

$$P_\delta = \frac{1}{50s+1}\begin{pmatrix} 0.878 & -0.864 \\ 1.082 & -1.096 \end{pmatrix}\begin{pmatrix} 1.2 & 0 \\ 0 & 0.8 \end{pmatrix}$$

Using same controller as before gives

$$P_\delta C = \frac{1}{s}\begin{pmatrix} 14.83 & -11.06 \\ 17.29 & -12.83 \end{pmatrix}$$

Example -closed loop step responses



$$PC = \frac{1}{s}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{versus} \qquad P_\delta C = \frac{1}{s}\begin{pmatrix} 14.83 & -11.06 \\ 17.29 & -12.83 \end{pmatrix}$$

With P: No interactions, nice step responses

With P_δ: Large interaction, 500 percent overshoot in step responses

Example -continued

The design is extremly sensitive to uncertainty on the inputs

But not to uncertainty on the outputs (easy to check)

Several indications of a directionality problem:

$$RGA(P) = \begin{pmatrix} 35 & -34 \\ -34 & 35 \end{pmatrix}$$

$$\mathrm{cond}(P) := \frac{\overline{\sigma}(P)}{\underline{\sigma}(P)} = 142$$

How do we analyse performance and robustness for MIMO systems ?

Nyquist for MIMO systems

There is a generalization of the Nyquist theorem to the MIMO case, see Maciejowski Ch. 2.8-2.10

$$G(s) = W(s)\Lambda(s)W^{-1}(s)$$

Characteristic loci: $\lambda_i(s) :=$ eigenvalues of $G(s)$

Theorem [MIMO Nyquist]: If $G(s)$ has P_o unstable poles, then the closed loop system with return ratio $-kG(s)$ is stable if the characteristic loci of $kG(s)$ encircle the point -1 P_o times anticlockwise.

Why not use characteristic loci ?

$G(s)$ can have well behaved char. loci with great apparent stability margins to -1, but the loop can still be quite non-robust

Example:

$G_0(s) = \frac{1}{s}I$ and $G_\delta(s) = \frac{1}{s}I + \delta \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$ have the same eigenvalues. Closed loops behave quite differently for large δ

Attempt of remedy: Try to acheive ”diagonally dominant” designs.

Performance and robustness is however best understood by using singular values instead of eigenvalues

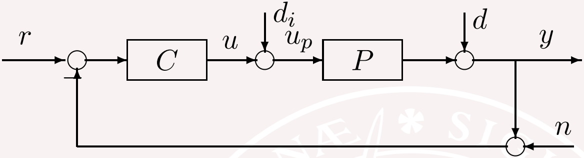
Performance and robustness

In the MIMO case the order of matrices matters

$$\begin{aligned} L_i &= CP, & L_o &= PC, \\ S_i &= (I + L_i)^{-1}, & S_o &= (I + L_o)^{-1}, \\ T_i &= I - S_i, & T_o &= I - S_o. \end{aligned}$$

TAT: Which of the following matrices are the same?

$$PC(I + PC)^{-1}, \quad C(I + PC)^{-1}P, \quad (I + PC)^{-1}PC$$
$$(I + CP)^{-1}CP, \quad P(I + CP)^{-1}C, \quad CP(I + CP)^{-1}$$



$$
\begin{aligned}
y &= T_o(r-n) + S_oPd_i + S_od, \\
r-y &= S_o(r-d) + T_on - S_oPd_i, \\
u &= CS_o(r-n) - CS_od - T_id_i, \\
u_p &= CS_o(r-n) - CS_od + S_id_i
\end{aligned}
$$

1) Good performance requires

$$\underline{\sigma}(L_o) >> 1, \quad \underline{\sigma}(C) >> 1.$$

2) Good robustness and good sensor noise rejection requires

$$\overline{\sigma}(L_o) << 1, \quad \overline{\sigma}(L_i) << 1, \quad \overline{\sigma}(C) \le M.$$

Conflict!!! Separate frequency bands!

MIMO requirements

Example: Weighted (output) sensitivity requirement

$$\|W_1(i\omega)S_o(i\omega)W_2(i\omega)\| \leq 1, \quad \forall\omega$$

$W_1(s), W_2(s)$: Rational function with no rhp poles or zeros

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The H_∞ norm = Induced L_2 norm

The H_∞ norm of a stable function $G(s)$ is given by

$$\|G\|_\infty = \sup_{\|u\|_2 \leq 1} \|Gu\|_2 = \sup_\omega \|G(j\omega)\|$$

(Parseval's relation + Theorem 4.3 in [Zhou+Doyle]).

For unstable $G(s)$ the norm is defined as infinite

$G \in RH_\infty^{p\times m}$ means $G(s)$ rational of size $p \times m$ with finite H_∞ norm

H_∞ norm computation

H_∞-norm computation requires a search

Theorem $\|C(sI-A)^{-1}B + D\|_\infty < \gamma$ if and only if

- $C(sI-A)^{-1}B$ is asymptotically stable
- $\sigma_{\max}(D) < \gamma$, hence $R = \gamma^2 I - D^* D > 0$
- H has no eigenvalues on the imaginary axis, where

$$H = \begin{pmatrix} A+BR^{-1}D^*C & BR^{-1}B^* \\ -C^*(I+DR^{-1}D^*)C & -(A+BR^{-1}D^*C)^* \end{pmatrix}$$

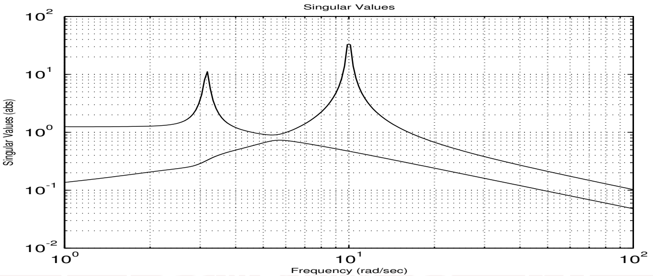
H_∞ norm computation - LMI alternative

$\|G\|_\infty < \gamma$ is equivalent to that there exists $P = P^* > 0$ such that

$$\begin{bmatrix} PA + A^*P & PB & C^* \\ B^*P & -\gamma I & D^* \\ C & D & -\gamma I \end{bmatrix} < 0$$

Linear Matrix Inequality. Convex optimization.

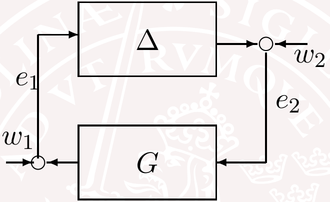
Singular value plot for 2×2 system



$$G(s) = \begin{bmatrix} \frac{10(s+1)}{s^2+0.2s+100} & \frac{1}{s+1} \\ \frac{s+2}{s^2+0.1s+10} & \frac{5(s+1)}{(s+2)(s+3)} \end{bmatrix}$$

The Matlab command `norm(G,'inf')` uses bisection together with the theorem above to get $\|G\|_\infty = 50.25$. Frequency sweep with 400 frequency points gives only the maximal value 43.53.

The Small Gain Theorem



Suppose $G \in RH_\infty^{p\times m}$. Then the closed loop system (G, Δ) is internally stable for all

$$\Delta \in \mathcal{B}RH_\infty := \{\Delta \in RH_\infty^{m\times p} \mid \|\Delta\|_\infty \le 1\}$$

if and only if $\|G\|_\infty < 1$.

Basic Uncertainty Models

Let $\mathcal{D}$ be a set of all allowable Δ's.

Additive uncertainty: $P_\Delta = P_0 + \Delta, \quad \Delta \in \mathcal{D}$.
Multiplicative uncertainty: $P_\Delta = (I + \Delta)P_0, \quad \Delta \in \mathcal{D}$.
Feedback uncertainty: $P_\Delta = P_0(I + \Delta P_0)^{-1}, \quad \Delta \in \mathcal{D}$.
Coprime factor uncertainty:

Let $P_0 = NM^{-1}$, M, $N \in RH_\infty$ and

$$P_\Delta = (N + \Delta_N)(M + \Delta_M)^{-1}, \quad \begin{pmatrix} \Delta_N \\ \Delta_M \end{pmatrix} \in \mathcal{D}.$$

TAT: Draw block diagrams for each of the uncertainty models!

Uncertainty models

- Very often

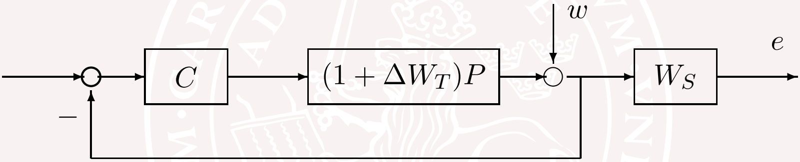
$$\mathcal{D} = \{W_1 \Delta W_2 \mid \|\Delta\|_\infty \le 1\}$$

  where W_1 and W_2 are given stable functions.
- The functions W_i provide the uncertainty profile. The main purpose of Δ is to account for phase uncertainty and to act as a scaling factor.
- Construction of uncertainty models is a nontrivial task

Robust Stability Tests

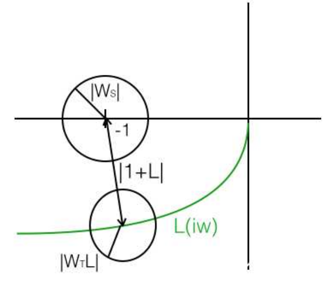
Uncertainty Model ($\|\Delta\| \leq 1$)	Robust stability test
$(I + W_1 \Delta W_2)P$	$\|W_2 T_o W_1\|_\infty < 1$
$P(I + W_1 \Delta W_2)$	$\|W_2 T_i W_1\|_\infty < 1$
$(I + W_1 \Delta W_2)^{-1} P$	$\|W_2 S_o W_1\|_\infty < 1$
$P(I + W_1 \Delta W_2)^{-1}$	$\|W_2 S_i W_1\|_\infty < 1$
$P + W_1 \Delta W_2$	$\|W_2 C S_o W_1\|_\infty < 1$
$P(I + W_1 \Delta W_2 P)^{-1}$	$\|W_2 S_o P W_1\|_\infty < 1$
$(\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$ $\Delta = [\tilde{\Delta}_N \; \tilde{\Delta}_M]$	$\left\| \begin{pmatrix} C \\ I \end{pmatrix} S_o \tilde{M}^{-1} \right\|_\infty < 1$
$(N + \Delta_N)(M + \Delta_M)^{-1}$ $\Delta = [\Delta_N \; \Delta_M]$	$\left\| M^{-1} S_i \begin{pmatrix} C & I \end{pmatrix} \right\|_\infty < 1$

Robust Performance - SISO case



Want $\|W_s S\| < 1$ for system with multiplicative uncertainty

Robust Performance - SISO case



Nominal Performance $\Leftrightarrow \|W_S S\|_\infty \le 1$

Robust Stability $\Leftrightarrow \|W_T T\|_\infty \le 1$

From figure:

$$\begin{aligned}\text{Robust Performance} &\Leftrightarrow |W_S| + |W_T L| \le |1+L|, \quad \forall s = i\omega \\ &\Leftrightarrow |W_S S| + |W_T T| \le 1, \quad \forall s = i\omega\end{aligned}$$

Robust Performance - SISO case

Robust Performance

$$\|T_{ew}\|_\infty < 1 \text{ for all } \|\Delta\| \le 1$$

is hence equivalent to the condition

$$\max_\omega \left[\underbrace{|W_S S|}_{\text{nominal performance}} + \underbrace{|W_T T|}_{\text{robust stability}} \right] < 1$$

RP almost guaranteed when we have NP + RS

$$\boxed{\text{NP + RS} \Rightarrow \text{RP/2}}$$

Explains why RP is not a big issue for SISO systems

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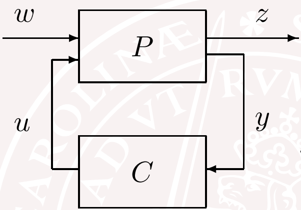
The H_∞ Optimization Problem



$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$$

$$T_{zw} = P_{11} + P_{12}C(I - P_{22}C)^{-1}P_{21}$$

Optimal control:

$$\min_{C-\text{stab}} \|T_{zw}\|_\infty$$

Suboptimal control: Given γ find stabilizing C such that

$$\|T_{zw}\|_\infty < \gamma \qquad \Longleftrightarrow \qquad \|z\|_2 < \gamma \|w\|_2, \quad \forall w$$

The optimal control problem is solved by iterating on γ

Understanding LQ control - completion of squares

If P satisfies the Riccati equation $A^TP + PA + Q - PBB^TP = 0$, then every solution to $\dot{x} = Ax + Bu$ with $\lim_{t\to\infty} x(t) = 0$ satisfies

$$\begin{aligned}
&\int_0^\infty [x^TQx + u^Tu]dt \\
&= \int_0^\infty |u + B^TPx|^2dt - 2\int_0^\infty (Ax + Bu)^TPxdt \\
&= \int_0^\infty |u + B^TPx|^2dt - 2\int_0^\infty \dot{x}^TPxdt \\
&= \int_0^\infty |u + B^TPx|^2dt - \int_0^\infty \frac{d}{dt}[x^TPx]dt \\
&= \int_0^\infty |u + B^TPx|^2dt + x(0)^TPx(0)
\end{aligned}$$

with the minimizing control law $u = -B^TPx$.

Understanding H_∞ control - completion of squares

If X satisfies the Algebraic Riccati Equation

$$A^T X + XA + Q - X(B_u B_u^T - B_w B_w^T/\gamma^2)X = 0$$

then $\dot{x} = Ax + B_u u + B_w w$ with $x(0) = 0$ gives

$$\int_0^\infty [x^T Q x + u^T u - \gamma^2 w^T w] dt$$
$$= \int_0^\infty |u + B_u^T X x|^2 dt - \gamma^2 \int_0^\infty |w - B_w^T X x|^2 dt$$

This can be viewed as a dynamic game between the player u, who tries to minimize and w who tries to maximize.

The minimizing control law $u = -B_u^T X x$ gives

$$\int_0^\infty [x^T Q x + u^T u] dt \le \gamma^2 \int_0^\infty w^T w dt$$

so the gain from w to $z = (Q^{1/2} x, u)$ is at most γ.

Riccati Equation for H_∞ Optimal State Feedback

Theorem: Consider $\dot{x} = Ax + B_u u + B_w w$, $x(0) = 0$, where (A, B_u) and (A, B_w) are stabilizable. Introduce the Hamiltonian

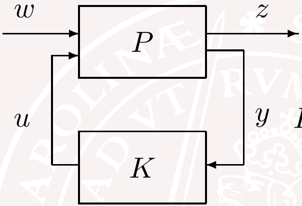
$$H_0 = \begin{pmatrix} A & B_w B_w^T/\gamma^2 - B_u B_u^T \\ -Q & -A^T \end{pmatrix}.$$

Then, the following conditions are equivalent:

1. There exists a stabilizing control law with $\int_0^\infty (x^T Q x + |u|^2) dt \leq \gamma^2 \int_0^\infty |w|^2 dt$
2. H_0 has no purely imaginary eigenvalues.

See [Zhou, p.237]

Output Feedback Assumptions



$$P = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & 0 & D_{zu} \\ C_y & D_{yw} & 0 \end{array}\right]$$

(A1) (A, B_w, C_z) is stabilizable and detectable,

(A2) (A, B_u, C_y) is stabilizable and detectable,

(A3) $D_{zu}^* \begin{pmatrix} C_z & D_{zu} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$,

(A4) $\begin{pmatrix} B_w \\ D_{yw} \end{pmatrix} D_{yw}^* = \begin{pmatrix} 0 \\ I \end{pmatrix}$.

State Space H_∞ optimization - DGKF formulas

The solution involves two AREs with Hamiltonian matrices

$$H_\infty = \begin{pmatrix} A & \gamma^{-2} B_w B_w^* - B_u B_u^* \\ -C_z^* C_z & -A^* \end{pmatrix}$$
$$J_\infty = \begin{pmatrix} A^* & \gamma^{-2} C_z^* C_z - C_y^* C_y \\ -B_w B_w^* & -A \end{pmatrix}$$

Theorem: There exists a stabilizing controller K such that $\|T_{zw}\|_\infty < \gamma$ if and only if the following three conditions hold:

1. $H_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $X_\infty = \mathrm{Ric}(H_\infty) \geq 0$,
2. $J_\infty \in \mathrm{dom}(\mathrm{Ric})$ and $Y_\infty = \mathrm{Ric}(J_\infty) \geq 0$,
3. $\rho(X_\infty Y_\infty) < \gamma^2$.

Moreover, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A}_\infty & -Z_\infty L_\infty \\ \hline F_\infty & 0 \end{array}\right]$$

where

$$\hat{A}_\infty = A + \gamma^{-2} B_w B_w^* X_\infty + B_u F_\infty + Z_\infty L_\infty C_y,$$

$$F_\infty = -B_u^* X_\infty, \; L_\infty = -Y_\infty C_y^*,$$

$$Z_\infty = (I - \gamma^{-2} Y_\infty X_\infty)^{-1}.$$

Furthermore, the set of all stabilizing controllers such that $\|T_{wz}\|_\infty < \gamma$ can be explicitly described (see [Zhou,p. 271]).

[Doyle J., Glover K., Khargonekar P., Francis B., *State Space Solution to Standard H^2 and H^∞ Control Problems*, IEEE Trans. on AC **34** (1989) 831–847.]

Matlab - General H_∞ design

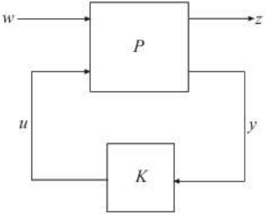
```
[K,CL,GAM,INFO] = hinfsyn(P,NMEAS,NCON)
```

`hinfsyn` computes a stabilizing H_∞ optimal `lti/ss` controller K for a partitioned `lti` plant P.

$$P = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix}$$

The controller, K, stabilizes the P and has the same number of states as P. The system P is partitioned where inputs to B_1 are the disturbances, inputs to B_2 are the control inputs, output of C_1 are the errors to be kept small, and outputs of C_2 are the output measurements provided to the controller. B_2 has column size (NCON) and C_2 has row size (NMEAS). The optional KEY and VALUE inputs determine tolerance, solution method and so forth.

The closed-loop system is returned in CL. This closed-loop system is given by `CL = lft(P,K)` as in the following diagram.



The achieved H_∞ cost γ is returned as GAM. The `struct` array INFO contains additional information about the design.

Matlab

```
[K,CL,GAM,INFO]=mixsyn(G,W1,W2,W3) or

mixsyn H-infinity mixed-sensitivity synthesis method for
robust control design. Controller K stabilizes plant G
and minimizes the H-infinity cost function
               ||   W1*S ||
               || W2*K*S ||
               ||   W3*T ||Hinf
where
    S := inv(I+G*K)          % sensitivity
    T := I-S = G*K/(I+G*K)  % complementary sensitivity
    W1, W2 and  W3 are stable LTI 'weights'
```

Matlab - mixsyn

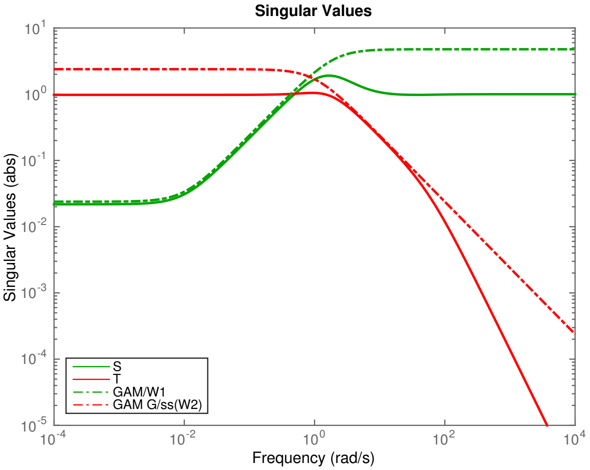
Minimizes H_∞ norm of

$$\begin{bmatrix} W_1(I+GK)^{-1} \\ W_2K(I+GK)^{-1} \\ W_3GK(I+GK)^{-1} \end{bmatrix}$$

```
[C,CL,GAM,INFO] = mixsyn(G,W1,W2,W3)
```

```
G = (s-1)/(s+1)^2;
W1 = 5*(s+2)/(100*s+1);
W2 = 0.1;
[K,CL,GAM] = mixsyn(G,W1,W2,[]);
L = G*K;
S = inv(1+L);
T = 1-S;
sigma(S,'g',T,'r',GAM/W1,'g-.',GAM*G/ss(W2),'r-.')
```

Result



Singular Values
Singular Values (abs)
Frequency (rad/s)
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-4}
10^{-2}
10^{0}
10^{2}
10^{4}
S
T
GAM/W1
GAM G/ss(W2)

Motor Control [Glad-Ljung Ex. 10.1]

Motor $P(s) = \dfrac{20}{s(s+1)}$.

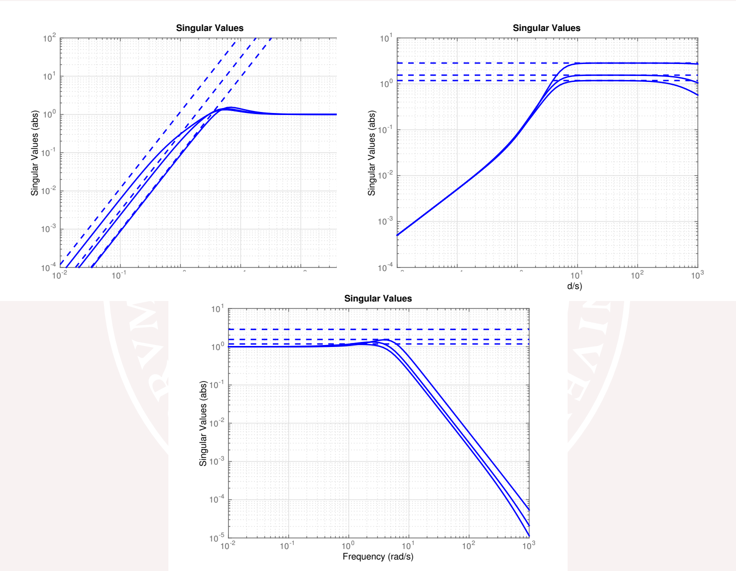
$$\text{Minimize } H_\infty \text{ norm of } \begin{bmatrix} W_1(I+PC)^{-1} \\ W_2C(I+PC)^{-1} \\ W_3PC(I+PC)^{-1} \end{bmatrix}, \quad \text{with} \quad \begin{array}{l} W_1 = \dfrac{k}{s^2} \\ W_2 = 1 \\ W_3 = 1 \end{array}$$

Increasing k gives higher bandwidth at the cost of larger controller gain

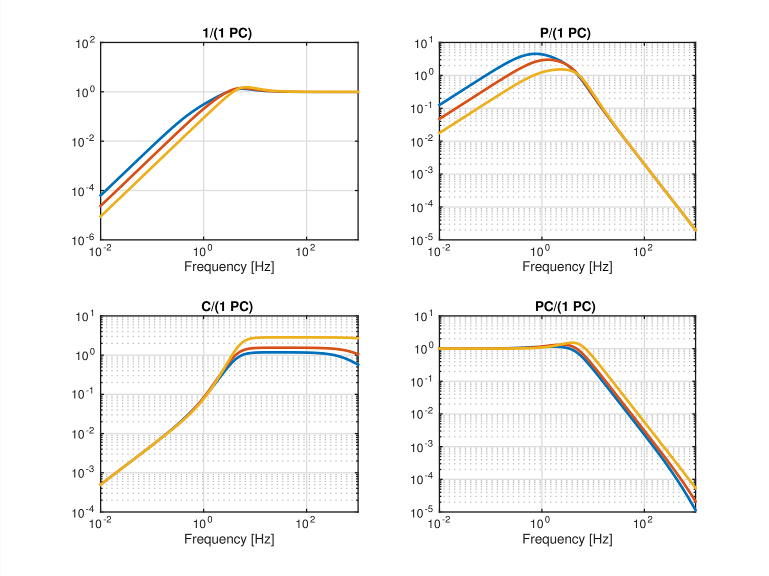
Shape of W_1 will enforce integral action. Try $k = 1, 5, 30$.

Needed to change to $P(s) = \dfrac{20}{(s+\epsilon)(s+1)}$ and $W_1 = \dfrac{k}{(s+\epsilon)^2}$

Result - S, KS and T for $k = 1, 5, 30$



Result - GangofFour for $k = 1, 5, 30$



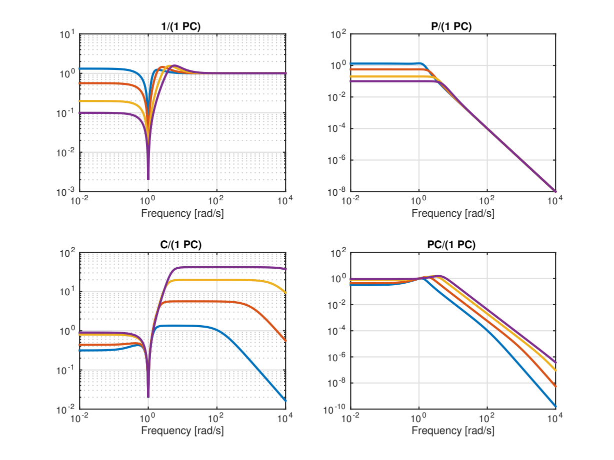
Resonant System

From earlier lectures and exercises:

$$P(s) = \frac{1}{s^2 + 0.02s + 1}$$

$$\text{Minimize } H_\infty \text{ norm of } \begin{bmatrix} W_1 S \\ W_2 CS \\ W_3 T \end{bmatrix}, \quad \text{where} \quad \begin{array}{l} W_1 = kP \\ W_2 = 1 \\ W_3 = [\,] \end{array}$$

Result - GangofFour for $k = 1, 10, 100, 420$

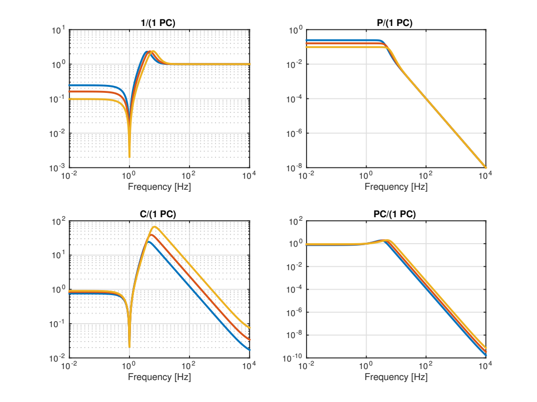


$k = 420$ gives $\|PS\|_\infty = 0.1$ and $\|CS\|_\infty = 42$
(similar value as our best design on the exercise)

Result - further optimization

Does the controller really need high gain beyond 10^4 rad/s?

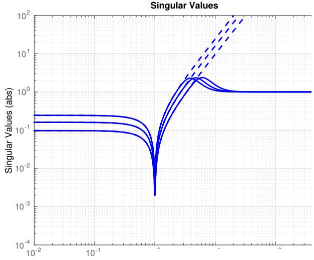
Change to $W_2 = s$ (or really $W_2 = (s + \epsilon)/(1 + \epsilon s)$) and retune k

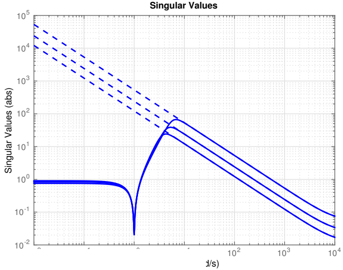


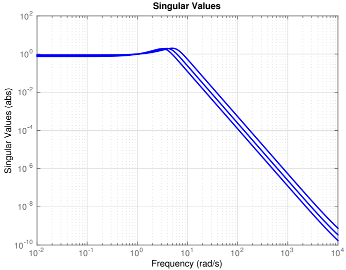
To achieve same $\|PS\|_\infty$ we need $\|CS\|_\infty = 67$, was 42.

Excellent reduction in noise power ($\|CS\|_2 = 126$, was 4200)

Result - S, KS and T





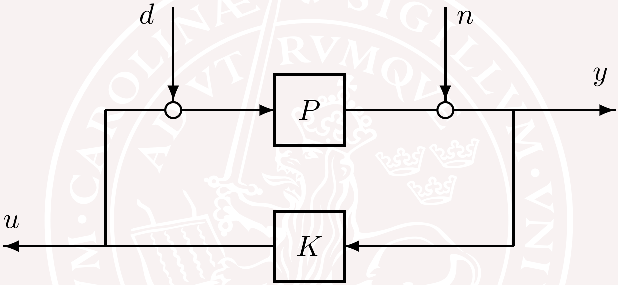


Robust Control

1. MIMO performance
2. Robustness and the H_∞-norm
3. H_∞-control
4. ν-gap metric
5. Glover-MacFarlane

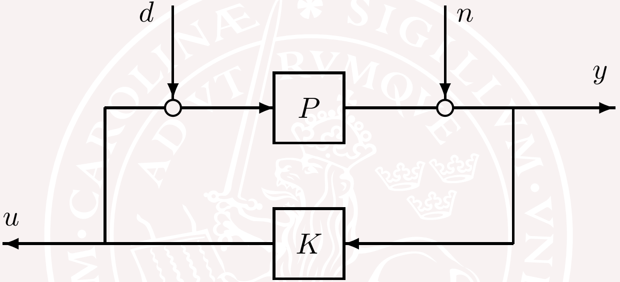
Theme: You get what you ask for!

The GOF matrix



$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \begin{bmatrix} n \\ d \end{bmatrix}$$

What is Good Performance?



What is captured by the norm of the GOF matrix?

$$\left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_{\infty} \quad ?$$

A Notion of Loop Stability Margin, $b_{P,K}$

A popular notion of stability margin is

$$b_{P,K} = \begin{cases} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty^{-1} & \text{if } K \text{ stabilizes } P \\ 0 & \text{otherwise} \end{cases}$$

The larger $b_{P,K} \in [0, 1]$ is, the more robustly stable the closed loop system is.

Remark: Note that $b_{P,K}$ depends on scalings of inputs and outputs.

Relation to Gain and Phase Margins

If P is a SISO plant and K a stabilizing controller then

$$\begin{aligned} \text{gain margin} &\geq \frac{1+b_{P,K}}{1-b_{P,K}}, \\ \text{phase margin} &\geq 2\arcsin(b_{P,K}). \end{aligned}$$

Proof: Exercise

ν-Gap Metric [Vinnicombe]

$$\delta_\nu(P_1, P_2) = \begin{cases} \|(I+P_2P_2^*)^{-\frac{1}{2}}(P_1-P_2)(I+P_1^*P_1)^{-\frac{1}{2}}\|_\infty \\ \text{if } \det(I + P_2^* P_1) \neq 0 \text{ on } jR \text{ and} \\ \text{wno} \det(I+P_2^*P_1) + \eta(P_1) = \overline{\eta}(P_2), \\ 1 \quad \text{otherwise} \end{cases}$$

where $\overline{\eta}$ (η) is the number of closed (open) RHP poles and wno is winding number.

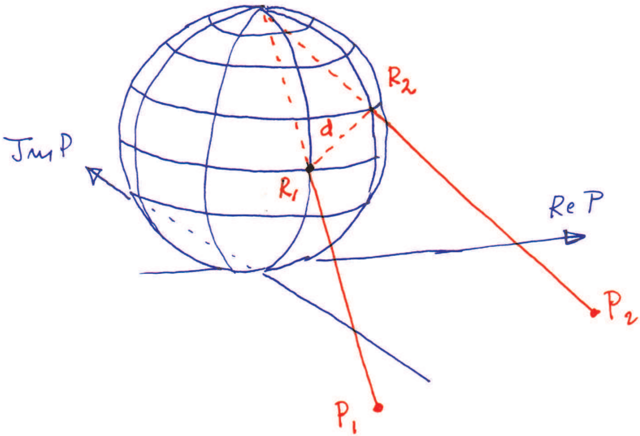
In scalar case it takes on the particularly simple form

$$\delta_\nu(P_1, P_2) = \sup_{\omega \in R} \frac{|P_1(j\omega) - P_2(j\omega)|}{\sqrt{1+|P_1(j\omega)|^2}\sqrt{1+|P_2(j\omega)|^2}}$$

whenever the winding number condition is satisfied.

Geometrical interpretation: Distance on the Riemann sphere

Geometric Interpretation



Example

Consider

$$P_1(s) = \frac{1}{s}, \quad P_2(s) = \frac{1}{s+0.1}.$$

We have $\|P_1 - P_2\|_\infty = +\infty$. However

$$\delta_\nu(P_1, P_2) \approx 0.0995$$

which means that the system are, in fact, very close.

```
[gap,nugap] = gapmetric(1/s,1/(s+0.1))
gap = 9.9510e-02
nugap = 9.9504e-02
```

Robustness Guarantees

Theorem For any P_0, P and K

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K} - \arcsin \delta_\nu(P_0, P).$$

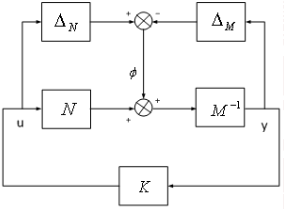
Corollary 1: If $b_{P_0,K} > \delta_\nu(P_0, P)$ then (P, K) is stable.

Corollary 2: For any P_0, P, K_0 and K

$$\arcsin b_{P,K} \geq \arcsin b_{P_0,K_0} - \arcsin \delta_\nu(P_0, P) - \arcsin \delta_\nu(K_0, K).$$

$b_{P,K}$ and Coprime Factor Uncertainty

Let $P = \tilde{M}^{-1}\tilde{N}$, where $\tilde{N}(i\omega)\tilde{N}(i\omega)^* + \tilde{M}(i\omega)\tilde{M}(i\omega)^* \equiv 1$. This is called *normalized coprime factorization*.



Large $b_{P,K}$ gives good robustness against coprime factor uncertainty

Computing Normalized Coprime Factors

Given $P(s) = C(sI - A)^{-1}B$, let Y be the stabilizing solution to

$$AY + YA^* - YC^*CY + BB^* = 0.$$

The matrix $A + LC$ is stable with $L := -YC^*$.

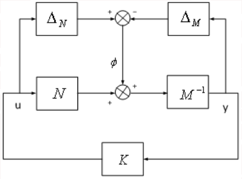
Lemma: A normalized factorization is given by

$$\begin{bmatrix}\tilde{N}(s) & \tilde{M}(s)\end{bmatrix} = \left(\begin{array}{c|cc} A + LC & B & L \\ \hline C & 0 & I \end{array}\right),$$

Proof: Denote $\mathcal{A}(s) = (sI - A + YC^*C)^{-1}$ and calculate

$$\begin{aligned}\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* &= I - C\mathcal{A}YC^* - CY\mathcal{A}^*C^* + C\mathcal{A}(B^*B + YC^*CY)\mathcal{A}^*C^* \\ &= I + C\mathcal{A}(B^*B + YC^*CY - Y(\mathcal{A}^*)^{-1} - \mathcal{A}^{-1}Y)\mathcal{A}^*C^* \\ &= I + C\mathcal{A}(\underbrace{B^*B - YC^*CY + AY + YA^*}_{=0})\mathcal{A}^*C^* = I\end{aligned}$$

$b_{P,K}$ and Coprime Factor Uncertainty

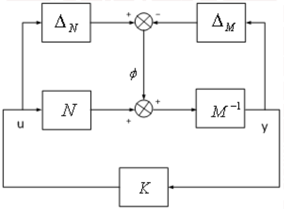


The process $P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$ in feedback with the controller K is stable for all $\Delta = [\tilde{\Delta}_N \ \tilde{\Delta}_M]$ with $\|\Delta\|_\infty \leq \epsilon$ iff

$$\left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty < \frac{1}{\epsilon} \qquad (1)$$

Finding K that minimizes LHS of (1) is an H_∞ problem.
Actually, no γ iteration needed!

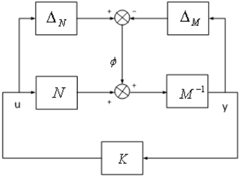
Proof of (1)



The interconnection of $P_\Delta = (\tilde{M} + \tilde{\Delta}_M)^{-1}(\tilde{N} + \tilde{\Delta}_N)$ and K can be rewritten as an interconnection of $\Delta = [\tilde{\Delta}_N \ \tilde{\Delta}_M]$ and

$$\begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1} \tilde{M}^{-1}$$

Proof of (1) - continued



The small gain theorem therefore gives the stability condition

$$\frac{1}{\epsilon} > \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1}\tilde{M}^{-1} \right\|_\infty$$
$$= \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1}\tilde{M}^{-1} \begin{bmatrix} \tilde{N} & \tilde{M} \end{bmatrix} \right\|_\infty$$
$$= \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I+PK)^{-1} \begin{bmatrix} I & P \end{bmatrix} \right\|_\infty$$

H_∞ Optimization of Normalized Coprime Factors

Theorem: Assume $Y \geq 0$ is the stabilizing solution to $AY + YA^* - YC^*CY + BB^* = 0$. Then $P = \tilde{M}^{-1}\tilde{N}$ is a normalized left coprime factorization and

$$\inf_{K-\text{stab}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (I + PK)^{-1}\tilde{M}^{-1} \right\|_\infty = \frac{1}{\sqrt{1 - \lambda_{\max}(YQ)}} =: \gamma_{opt}$$

where $Q(A - YC^*C) + (A - YC^*C)^*Q + C^*C = 0$.
Moreover, a controller achieving $\gamma > \gamma_{opt}$ is

$$\begin{aligned} K(s) &= \left(\begin{array}{c|c} A - BB^*X_\infty - YC^*C & -YC^* \\ \hline -B^*X_\infty & 0 \end{array} \right) \\ X_\infty &= \frac{\gamma^2}{\gamma^2 - 1} Q \left(I - \frac{\gamma^2}{\gamma^2 - 1} YQ \right)^{-1} \end{aligned}$$

Note: No iteration of γ needed.

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Loop-Shaping Design

A good performance controller design typically requires

- large gain in the low frequency region:

$$\underline{\sigma}(PK) >> 1, \quad \underline{\sigma}(KP) >> 1, \quad \underline{\sigma}(K) >> 1.$$

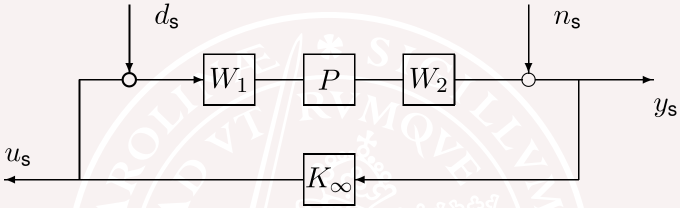
- small gain in the high frequency region:

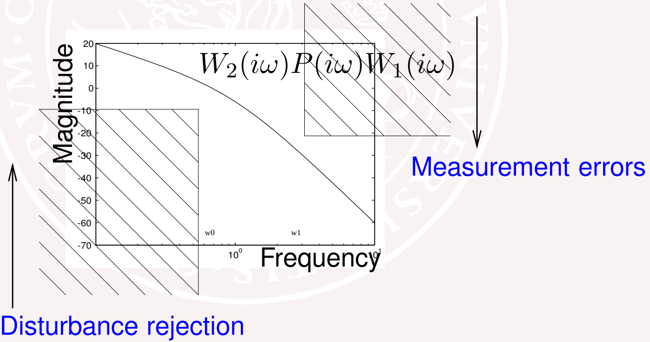
$$\overline{\sigma}(PK) << 1, \quad \overline{\sigma}(KP) << 1, \quad \overline{\sigma}(K) \leq M$$

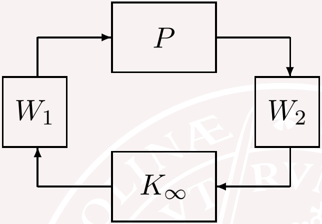
where M is not too large.

Wouldn't it be nice to be able to do loopshaping worrying only about the gains and not care about phase and stability ?

Glover McFarlane Loopshaping







1) Choose W_1 and W_2 and absorb them into the nominal plant P to get the shaped plant $P_s = W_2 P W_1$.
2) Calculate $b_{opt}(P_s)$. If it is small (< 0.25) then return to Step 1 and adjust weights.
3) Select $\epsilon \leq b_{opt}(P_s)$ and design the controller K_∞ such that

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty < \frac{1}{\epsilon}.$$

4) The final controller is $K = W_1 K_\infty W_2$.

Matlab - loopsyn

Syntax

```
[K,CL,GAM,INFO]=loopsyn(G,Gd)
[K,CL,GAM,INFO]=loopsyn(G,Gd,RANGE)
```

Description

`loopsyn` is an H_∞ optimal method for loopshaping control synthesis. It computes a stabilizing H_∞ controller K for plant G to shape the `sigma` plot of the loop transfer function GK to have desired loop shape G_d with accuracy γ = GAM in the sense that if ω_0 is the 0 db crossover frequency of the `sigma` plot of $G_d(j\omega)$, then, roughly,

$\underline{\sigma}(G(j\omega)K(j\omega)) \geq \frac{1}{\gamma}\,\underline{\sigma}(G_d(j\omega))$ for all $\omega < \omega_0$	**(1-14)**

$\overline{\sigma}(G(j\omega)K(j\omega)) \leq \gamma\,\overline{\sigma}(G_d(j\omega))$ for all $\omega > \omega_0$	**(1-15)**

The STRUCT array `INFO` returns additional design information, including a MIMO stable min-phase shaping pre-filter W, the shaped plant $G_s = GW$, the controller for the shaped plant $K_s = WK$, as well as the frequency range $\{\omega_{min},\omega_{max}\}$ over which the loop shaping is achieved

Input Argument	Description
G	LTI plant
Gd	Desired loop-shape (LTI model)
RANGE	(optional, default `{0,Inf}`) Desired frequency range for loop-shaping, a 1-by-2 cell array $\{\omega_{min},\omega_{max}\}$; ω_{max} should be at least ten times ω_{min}

Remarks:

- In contrast to the classical loop shaping design we do not treat explicitly closed loop stability, phase and gain margins. Thus the procedure is simple.
- Observe that

$$\left\| \begin{bmatrix} I \\ K_\infty \end{bmatrix} (I + P_s K_\infty)^{-1} \tilde{M}_s^{-1} \right\|_\infty = \left\| \begin{bmatrix} W_2 \\ W_1^{-1} K \end{bmatrix} (I + PK)^{-1} \begin{bmatrix} W_2^{-1} & PW_1 \end{bmatrix} \right\|_\infty$$

  so it has an interpretation of the standard H_∞ optimization problem with weights.
- BUT!!! The open loop under investigation on Step 1 is $K_\infty W_2 P W_1$ whereas the actual open loop is given by $W_1 K_\infty W_2 P$ and $P W_1 K_\infty W_2$. This is not really what we has shaped!

Thus the method needs validation in the MIMO case.

Justification of H_∞ Loop Shaping

We show that the degradation in the loop shape caused by K_∞ is limited. Consider low-frequency region first.

$$\begin{aligned}\underline{\sigma}(PK) &= \underline{\sigma}(W_2^{-1}P_sK_\infty W_2) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_2)},\\ \underline{\sigma}(KP) &= \underline{\sigma}(W_1K_\infty P_sW_1^{-1}) \geq \frac{\underline{\sigma}(P_s)\underline{\sigma}(K_\infty)}{\kappa(W_1)}\end{aligned}$$

where κ denotes conditional number. Thus small $\underline{\sigma}(K_\infty)$ might cause problem even if P_s is large. Can this happen?

Theorem: Any K_∞ such that $b_{P_s,K_\infty} \geq 1/\gamma$ also satisfies

$$\underline{\sigma}(K_\infty) \geq \frac{\underline{\sigma}(P_s) - \sqrt{\gamma^2-1}}{\sqrt{\gamma^2-1}\,\underline{\sigma}(P_s)+1} \quad \text{if } \underline{\sigma}(P_s) > \sqrt{\gamma^2-1}.$$

Corollary: If $\underline{\sigma}(P_s) >> \sqrt{\gamma^2-1}$ then $\underline{\sigma}(K_\infty) \geq 1/\sqrt{\gamma^2-1}$

Consider now high frequency region.

$$\begin{aligned}\overline{\sigma}(PK) &= \overline{\sigma}(W_2^{-1}P_sK_\infty W_2) \leq \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_2),\\ \overline{\sigma}(KP) &= \overline{\sigma}(W_1K_\infty P_sW_1^{-1}) \leq \overline{\sigma}(P_s)\overline{\sigma}(K_\infty)\kappa(W_1).\end{aligned}$$

Can $\overline{\sigma}(K_\infty)$ be large if $\overline{\sigma}(P_s)$ is small?

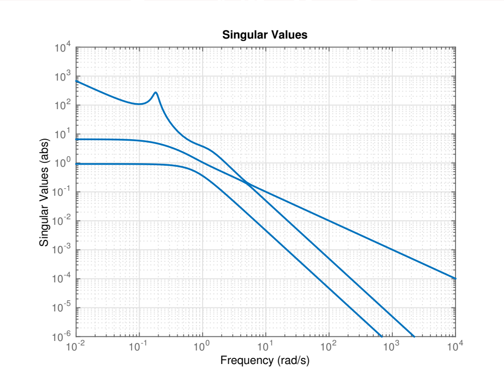
Theorem: Any K_∞ such that $b_{P_s,K_\infty} \geq 1/\gamma$ also satisfies

$$\overline{\sigma}(K_\infty) \leq \frac{\sqrt{\gamma^2-1}+\overline{\sigma}(P_s)}{1-\sqrt{\gamma^2-1}\,\overline{\sigma}(P_s)} \quad \text{if } \overline{\sigma}(P_s) < \frac{1}{\sqrt{\gamma^2-1}}.$$

Corollary: If $\overline{\sigma}(P_s) << 1/\sqrt{\gamma^2-1}$ then $\overline{\sigma}(K_\infty) \leq \sqrt{\gamma^2-1}$

Example: Vertical Aircraft Dynamics [Glad/Ljung]

See LQG lecture (3 inputs, 3 outputs: height, forward speed, pitch)

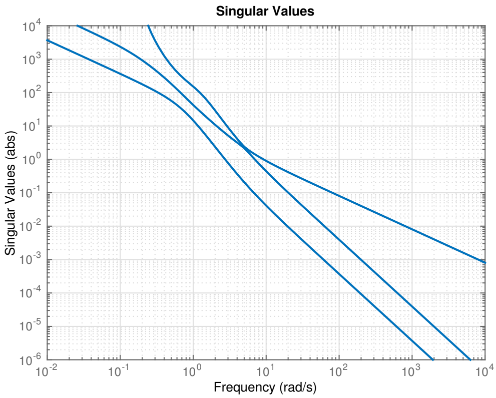


Open loop singular values

Will try to achieve bandwidth 10 rad/s, following Glad/Ljung.

Will need control gain about 250 in some direction at 10 rad/s

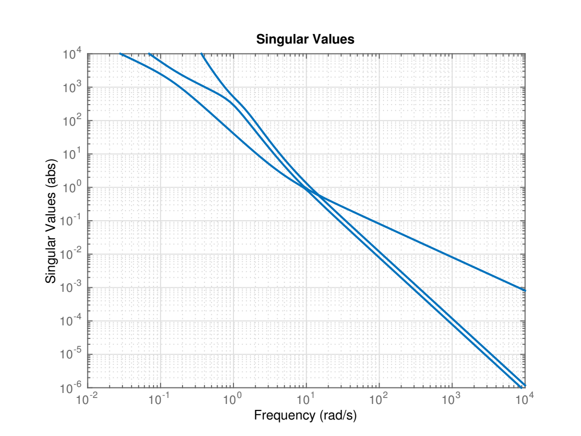
Example: Vertical Aircraft Dynamics



Start by adding diagonal PI-controllers

Singular values for GW_1 with $W_1 = 8(1 + \frac{5}{s})I_3$

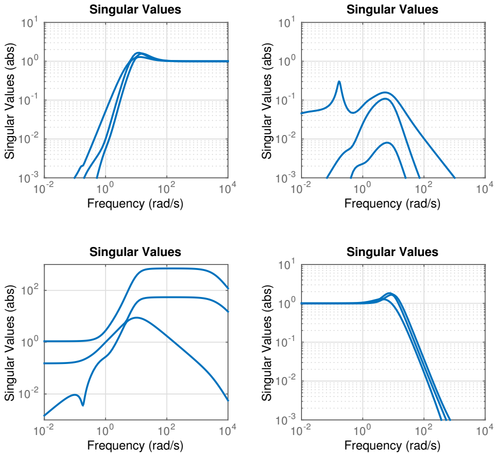
Example: Vertical Aircraft Dynamics



Squeeze singular values together near 10 rad/s

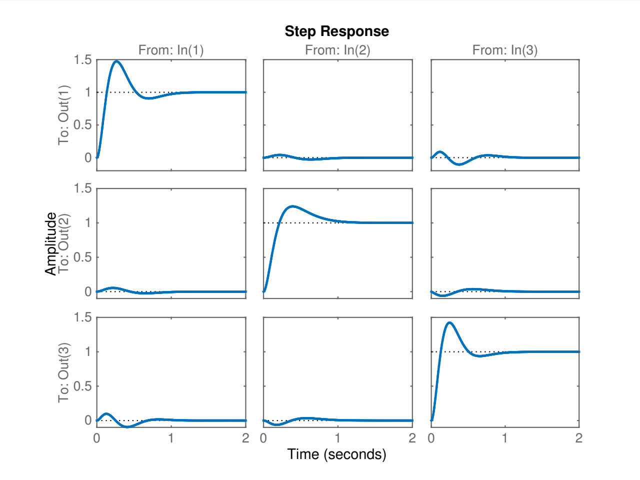
$W_1 := W_1(I + k\mathrm{Re}(v_3 v_3^*))$ where $GW_1(i10) = U\Sigma V^*$, $v_3 = V(:,3)$

Vertical Aircraft Dynamics - Gang of Four



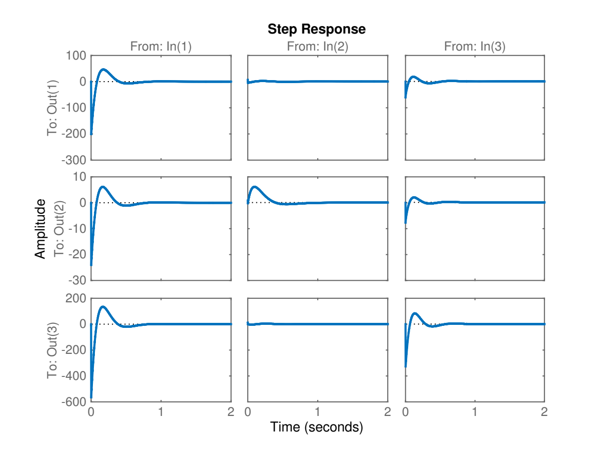
S and T as wanted. Control gain quite high, see KS

Example: Vertical Aircraft Dynamics



Nice decoupled step responses. Time constant about 0.1s.

Example: Vertical Aircraft Dynamics



Large control signals needed to move 1 meter in 0.1s.

Too aggressive design with 10 rad/s bandwidth? Good start for futher tuning.

Handin

Do one of the two KTH laborations, see home page

- H_∞ design (easiest)
- Dynamic Decoupling and Glover-McFarlane design