Requirements

Bo Bernhardsson and Karl Johan Åström

Department of Automatic Control LTH,
Lund University

Requirements and Limitations

1. Introduction
2. The basic feedback system
3. A broad view of control system design
4. Command signal following - System inversion
5. Disturbances
6. Process uncertainty
7. Robustness
8. Summary

Theme: Requirements for design and verification

Introduction

The purpose of this lecture is to build up the machinery to understand and formulate the requirements for control system design. There are two main issues: performance and robustness. We will cover

- System Architecture
- Feedback Fundamentals
- Fundamental limitations
- Broad understanding of design issues
  - Insight and understanding
  - Trade-offs
- How to capture essential system properties
- Parameters that capture requirements
- Design parameters

Control Requirements

- Requirements are key, they should drive design
- Choose requirements wisely and early in the design
- Many possibilities, make a sensible choice and use it systematically
- Check requirements automatically at each stage in the design process by simulation (SIL) or hardware in the loop simulation (HIL)
- Design equipment so that requirements can be checked experimentally
- Provide access points and measurement points
- Steps and ramps for time responses, chirp signals to measure frequency response

Design Issues

Disturbances

- Effect of feedback on disturbances
- Attenuate effects of load disturbances
- Moderate measurement noise injection

Robustness

- Reduce effects of process variations
- Reduce effects of modeling errors

Command signal response

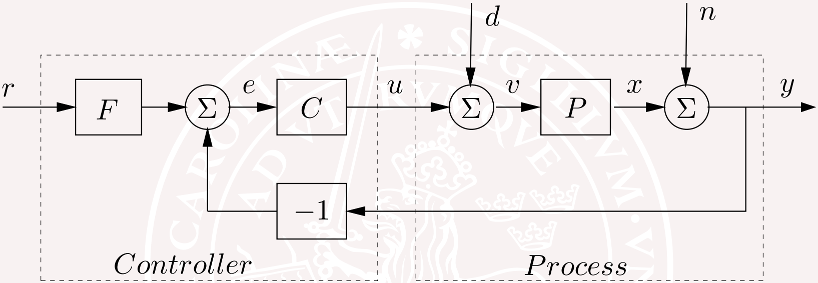
- Follow command signals
- Architectures with two degrees of freedom (2DOF)

Requirements

1. Introduction
2. The basic feedback system
3. A broad view of control system design
4. Command signal following - System inversion
5. Disturbances
6. The sensitivity functions - Fundamental limitations
7. Process uncertainty
8. Robustness
9. Summary

Theme: Requirements for design and verification

A Basic Feedback System

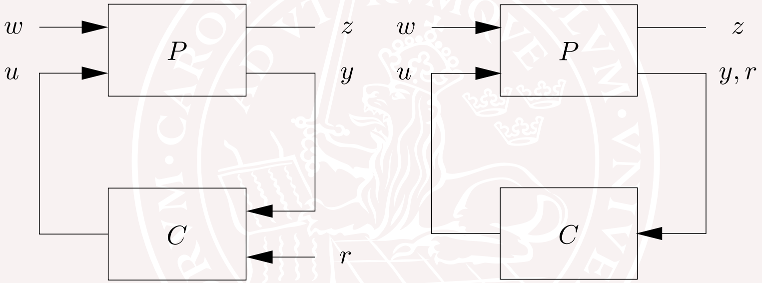


Ingredients:

- Controller: feedback C, feedforward F
- Load disturbance d : Drives the system from desired state
- Process: transfer function P
- Measurement noise n : Corrupts information about x
- Process variable x should follow reference r

A More General Setting

Load disturbances need not enter at the process input and measurement noise may also enter in different way. More general structures are.



$$w = (d, n),\ z = (e, v, \ldots) \qquad w = (d, n, r),\ z = (e, v, \ldots)$$

These problems can be dealt with in the same way but we stick to the simpler case. In practice always useful to understand the nature of the disturbances.

Typical Requirements

A controller should

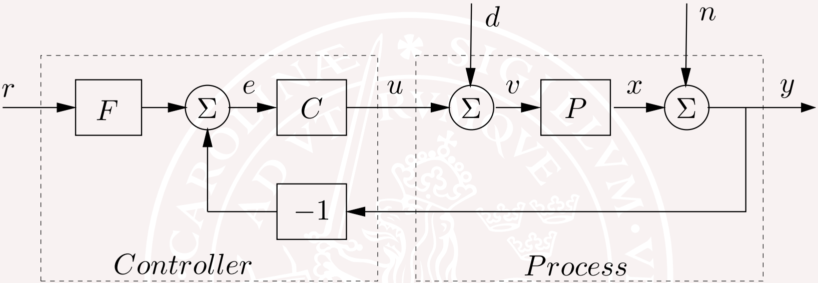
- **A:** Reduce effects of load disturbances
- **B:** Do not inject too much measurement noise into the system
- **C:** Make the closed loop insensitive to variations in the process
- **D:** Make output follow command signals well

Systems with two degrees of freedom

- Design feedback for A, B and C
- Then design feed-forward to handle D

Systems with error feedback do not allow this separation of responses to command signal and disturbances.

Architecture with Two Degrees of Freedom



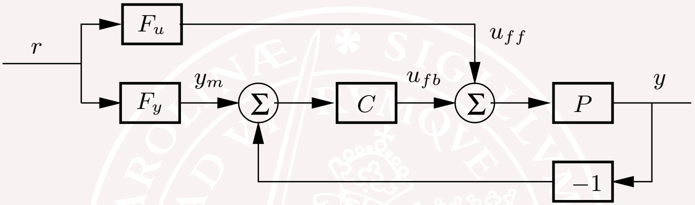
Horowitz Synthesis of Frieedback Systems 1963: Design the feedback C to achieve

- Low sensitivity to load disturbances d
- Low injection of measurement noise n
- High robustness to process uncertainty and process variations

Design the feedforward F to achieve

- Desired response to command signals r

Other Architectures with 2DOF



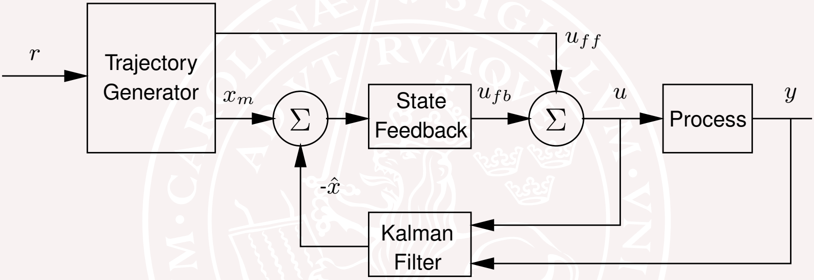
For linear systems all 2DOF configurations have the same properties. For the systems above we have $CF = F_u + CF_y$

PID Control - Setpoint weighting

$$u(t) = k_p(\beta y_{sp}(t) - y_f(t)) + k_i \int_0^t (y_{sp}(\tau) - y_f(\tau))d\tau - k_d\left(\gamma \frac{dy_{sp}}{dt} - \frac{dy_f}{dt}\right)$$

- Tune k_p, k_i, k_d and filtering ($y_f = G_f y$) for load disturbances, robustness and measurement noise
- Tune β and γ for set point response

State Feedback - Kalman Filter Architecture

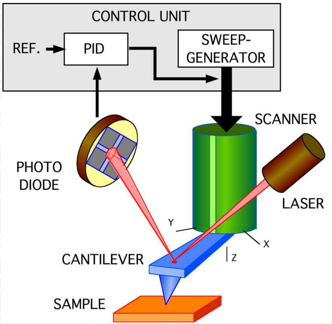


- A nice separation of the different functions
- The signals x_m and u_{ff} can be generated from r in real time or from stored tables (robotics)

Some Systems only Allow Error Feedback

Disk drive

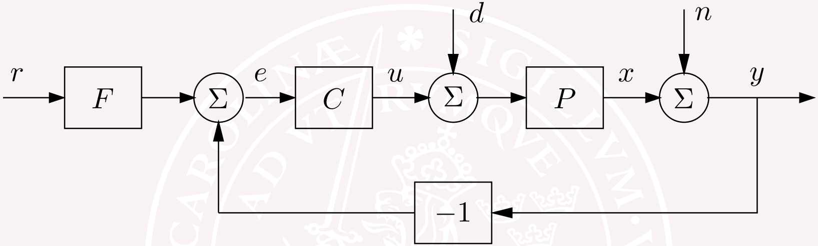
Atomic Force Microscope



Only error can be measured

Design for command disturbance attenuation and command signal response can not be separated!

The Gangs of Four and Seven



$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$E = -\frac{P}{1+PC}D - \frac{1}{1+PC}N + \frac{F}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four GoF)

$$\frac{PC}{1+PC} \qquad \frac{P}{1+PC} \qquad \frac{C}{1+PC} \qquad \frac{1}{1+PC}$$

- The system with a controller having two degrees of freedom is characterized by *seven* transfer function (The Gang of Seven GoS)

$$\frac{PCF}{1+PC} \qquad \frac{CF}{1+PC} \qquad \frac{F}{1+PC}$$

- To fully understand a system it is necessary to look at all transfer functions
- It may be strongly misleading to only show properties of a few systems for example the response of the output to command signals, a common omission in literature.

The Gangs of Four and Seven

Response of y to load disturbance d is characterized by

$$\frac{P}{1+PC}$$

Response of u to measurement noise n is characterized by

$$\frac{C}{1+PC}$$

Robustness to process variations is characterized by

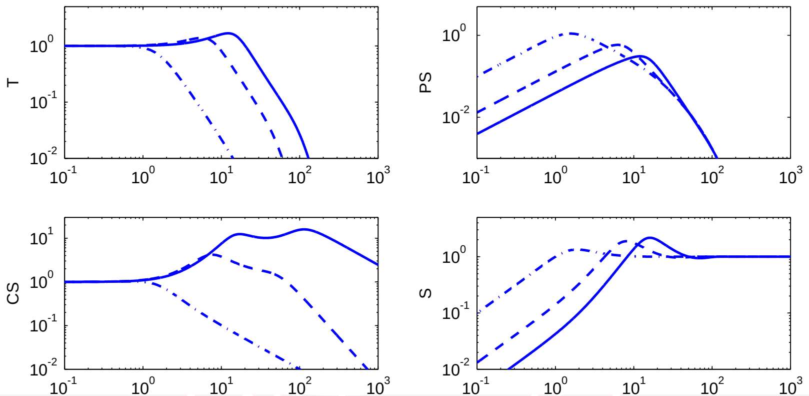
$$S = \frac{1}{1+PC}, \qquad T = \frac{PC}{1+PC}, \qquad S+T=1$$

Responses of y, u and e to reference signal r are characterized by

$$\frac{PCF}{1+PC}, \qquad \frac{CF}{1+PC}, \qquad \frac{F}{1+PC}$$

Requirements are based on features of the Gangs

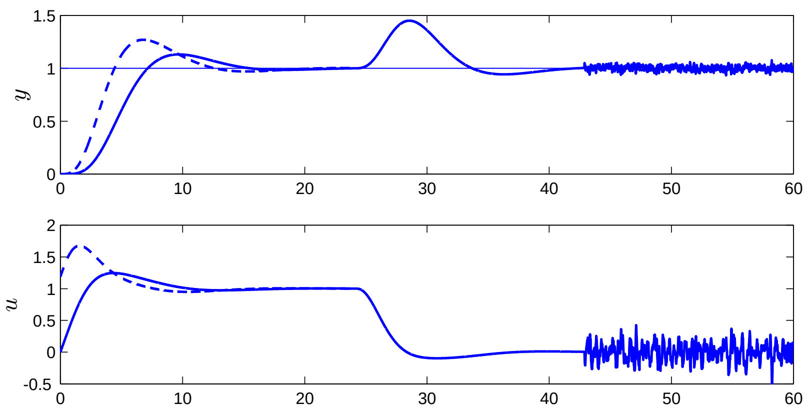
Gain Curves of the Gang of Four



Gain curves of the Gang of Four for a heat conduction process with I (dash-dotted), PI (dashed) and PID (full) controllers.

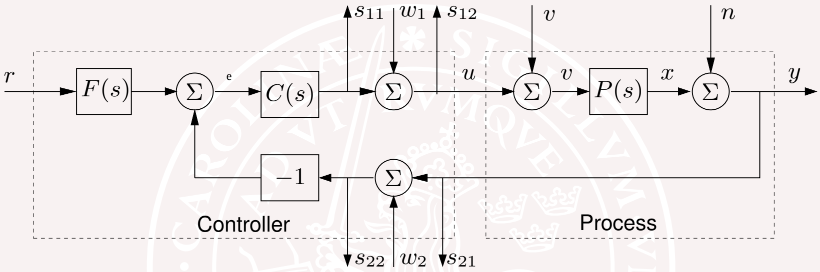
One plot gives a good overview of performance and robustness!

Time Response of the Gang of Seven



Error feedback (dashed) set point weighting (full)
One plot gives a good overview!

Testing Requirements



S Change w_1 measure s_{12} or change w_2 measure s_{22}

T Change w_1 measure s_{11} or change w_2 measure s_{21}

G_{yv} Change w_1 measure s_{21}

G_{un} Change w_2 measure s_{11}

TF Change y_{sp} measure s_{21}

SFC Change y_{sp} measure s_{11}

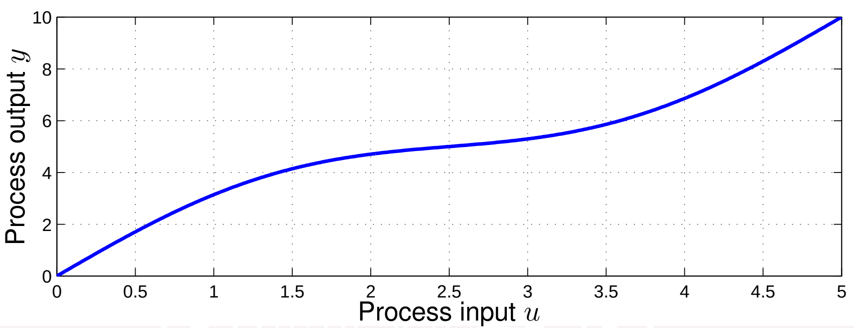
Requirements

Theme: Requirements for design and verification

Key Issues

- Understand the process, its static and dynamic properties
  - Possible equilibria, wide sense controllable
  - Experiments and physical models are useful
  - Stable, unstable, integrators
- Large signal behavior
  - Rate and level saturation of actuators
- Small signal behavior
  - Noise, AD and DA quantization, friction
- Dynamics limitations (more later)
  - RHP poles and zeros
  - Time delays
- The cardinal sin is to believe that the process is given!
  - Move or add sensors (changes zeros)
  - Change the process

Static Characteristics



- Start by exploring the static input-output relation practically and theoretically. Sweep input up and down, look for hysteresis.
- Information about signal levels and variations in static gain
- Robust or gain scheduling

Large Signal Behavior

- Process may be strongly nonlinear
- Control signals and their rates are often limited (JAS)
- Actuator and drive amplifier selection
- Limitations on response time and bandwidth
- Optimal control theory and algorithms are very useful
- Make sure you understand the problem and constraints!
- Tools: Optimal control

  Time optimal control - Tell what is possible
  Optimize energy or other criteria
  Grundelius Thesis 2001 #62 - Move quickly and avoid sloshing
  minimum time + minimum energy

Voice coil drive for a hard disk drive

$$J\frac{d^2\varphi}{dt^2} = T = k_t\, r\, I$$

$$m\frac{d^2x}{dt^2} = F = k_t I$$

$$r = 0.05\ m$$

$$J = 5 \times 10^{-6}\ kg\, m^2$$

$$m = 2 \times 10^{-3}\ kg$$

$$k_t = 2\ N/A$$

$$I_{max} = 0.5A$$

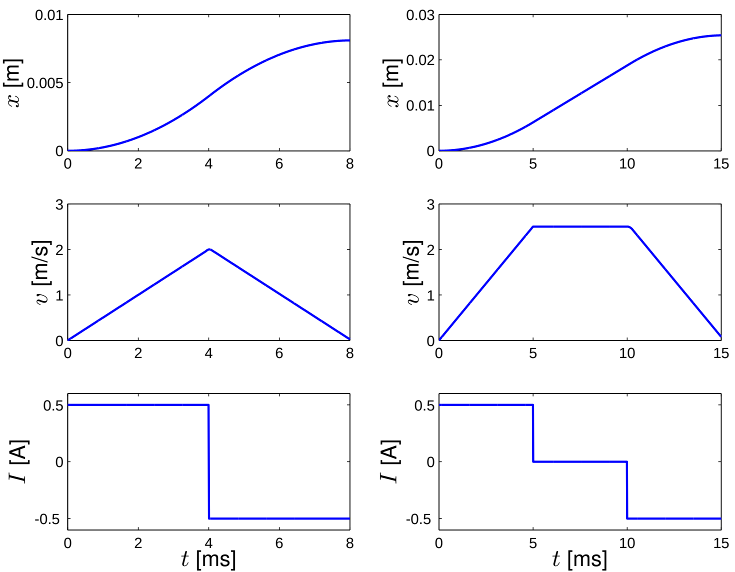
$$V_{max} = 5\ V$$

Maximum acceleration $a_{max} = \frac{k_t I_{max}}{m} = 500\ m/s^2$ ($\approx 50g$)

Maximum velocity $v_{max} = \frac{V_{max}}{k_t} = 2.5$ m/s

http://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=1003997

$a_{max} = 500\ m/s^2$, $v_{max} = 2.5\ m/s$



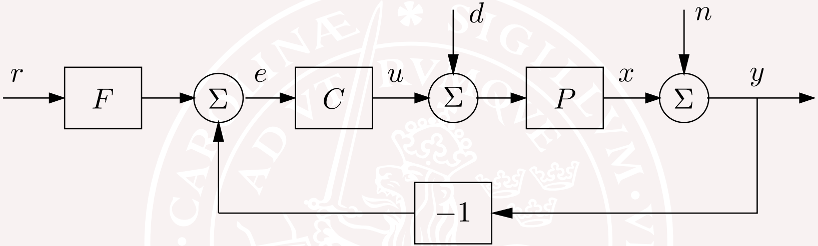
Small Signal Behavior

Important factors:

- Noise
- Sensor resolution
- Resolution of AD converter
- Resolution of DA converter
- Friction

The consequences are typically small variations around the equilibrium which limits the achievable precision

Measurement Noise



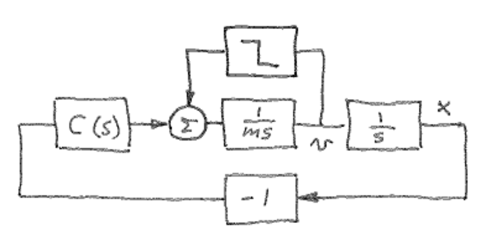
Measurement noise typically dominated by high frequencies.

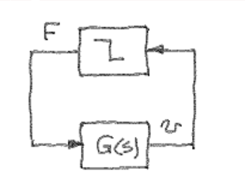
TAL: What does it mean if there is a bias?

$$G_{un}(s) = -\frac{C}{1+PC} = -SC, \qquad G_{yn} = \frac{PC}{1+PC} = T$$

G_{un} is the important transfer function.

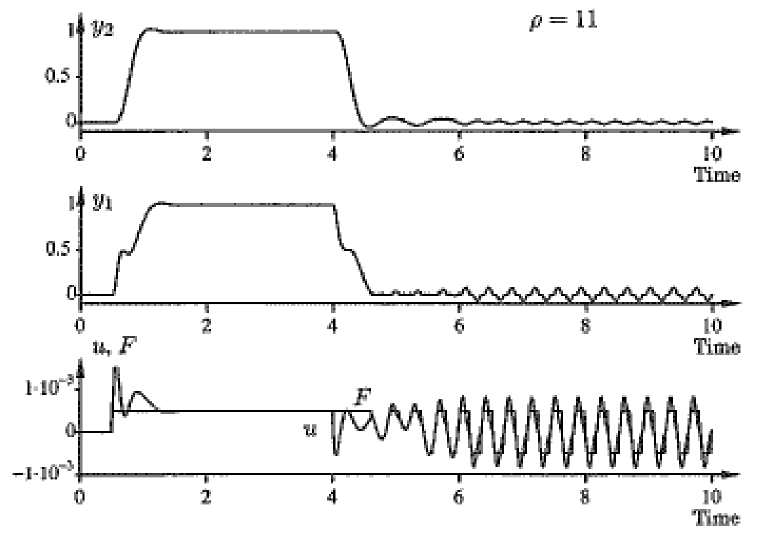
Friction





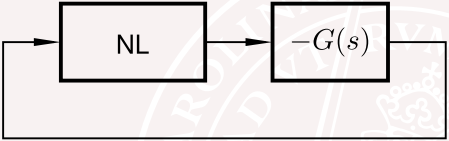
- Coulomb friction model
- System oscillates with the frequency where $G(i\omega)$ has a phase lag of 180°
- Exact analysis possible but more complicated
- There are many friction models

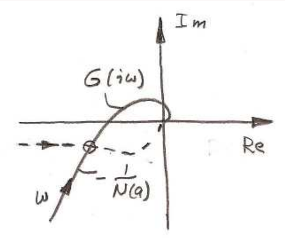
Friction generated oscillations



Olsson+kj IEEE Trans CST **9**(2001) 629-636

Describing Function Analysis - Harmonic Balance

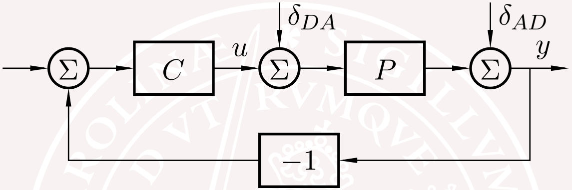




Approximate output of non-linearity with first harmonics. Propagation of the first harmonic can be described by the function $N(a)$ where a is the amplitude of the sinusoidal input. Tracing signals around the loops give the following condition for oscillation: $N(a)G(i\omega) = -1$. Describing function for relay: $N_{relay}(a) = 4d/(\pi a)$. Locus of $-1/N_{relay}(a)$ is the negative real axis.

TAL: Does the intersection represent a stable oscillation?

AD and DA Resolution - Noise Approximation

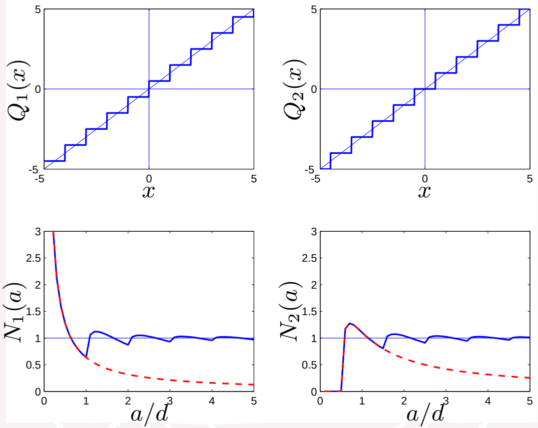


Approximate round-off errors by high frequency noise with variance $\sigma_\delta^2 = \frac{\delta^2}{12}$

$$Y = \frac{1}{1+PC}\delta_{AD} + \frac{P}{1+PC}\delta_{DA}$$
$$U = -\frac{C}{1+PC}\delta_{AD} - \frac{PC}{1+PC}\delta_{DA}$$

Make an estimate by using the gains of the transfer functions at the frequency obtaind by describing function analysis

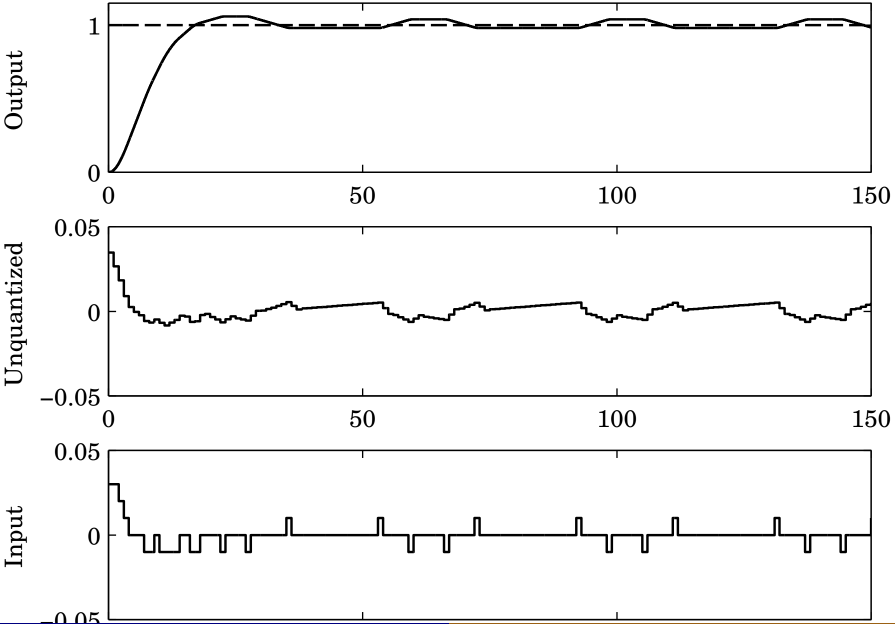
Describing Functions for Round-off



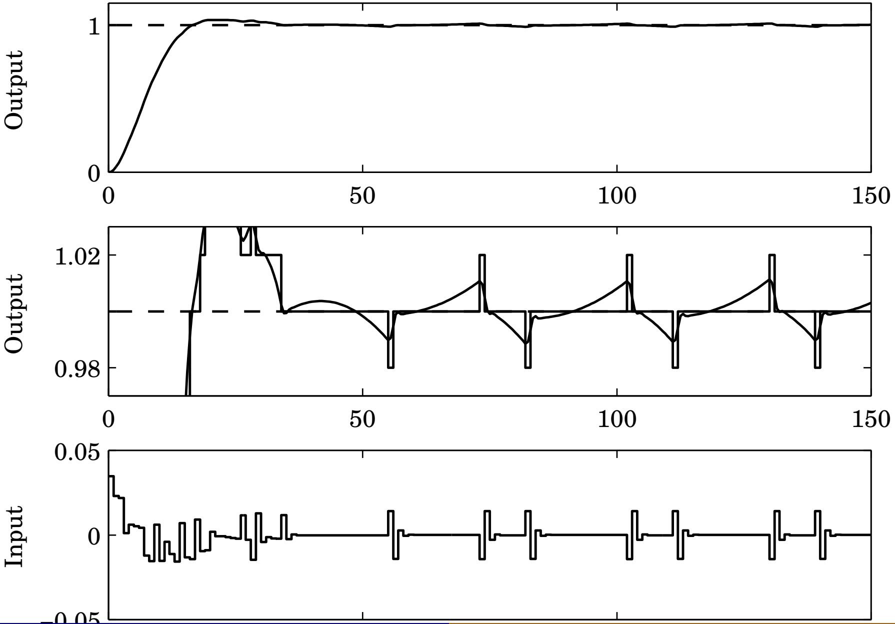
$$N_{relay}(a) = 2d/(\pi a)$$

$$N_{dead-zone}(a) = \begin{cases} 0 & a < d/2 \\ 4d\sqrt{1 - d^2/(2a)^2}\,/(\pi a) & a \geq d/2 \end{cases}$$

Computer Control of Double Integrator DA Quantization

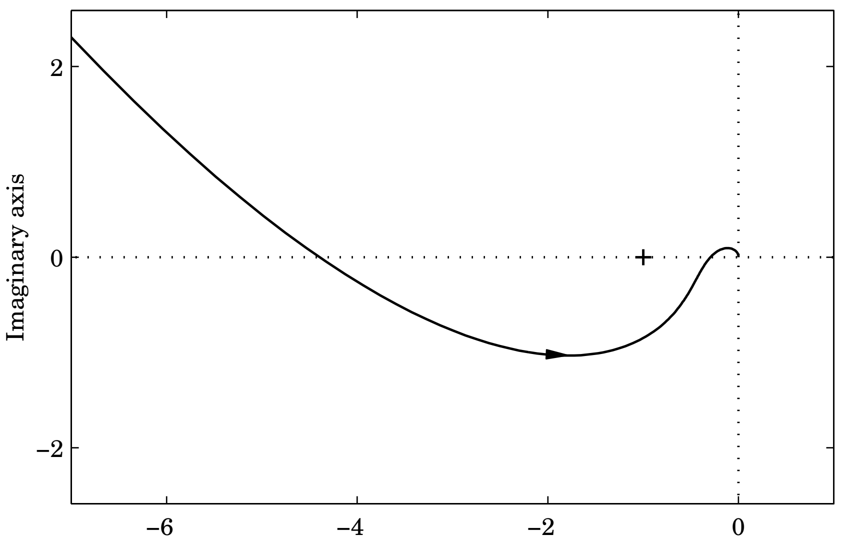


Computer Control of Double Integrator AD Quantization



Describing Function Analysis

Nyquist curve for computer control of double integrator with quantization in DA converter

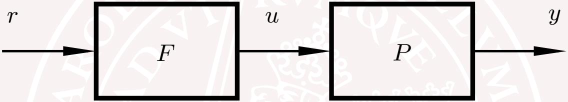


Requirements

Theme: Requirements for design and verification

Command Signal Following - System Inversion

Find a stable feedforward controller F that combined with the process P gives a desired relation from reference r to output y



Hence

$$G_m = PF, \qquad F = P^{-1}G_m$$

Feedforward design requires system inversion!

- To avoid differentiation: Pole excess of G_m equal to or greater than pole excess of P
- Fast response requires large signals and large signal rates
- Time delays and RHP zeros cannot be inverted

Difficulties with System Inversion

Let $P(s) = \dfrac{1}{s+1}$ and $G_m(s) = 1$, then $F(s) = s + 1$. The control signal is $u(t) = r(t) + \dfrac{dr(t)}{dt}$, which can be very large if the reference signal changes rapidly. The control signal is infinite for a step input.

Let $P(s) = \dfrac{1}{s+1}$ and $G_m(s) = \dfrac{a}{s+a}$, then $F(s) = \dfrac{a(s+1)}{s+a}$. For rapidly changing control signals we have $u(t) \approx ar(t)$. The control signal is then large if we require a fast response (large a).

The inverse is always stable for processes with minimum phase dynamics, the achievable performance is limited by limitations on the control signal.

Processes with right half plane zeros have unstable inverses, processes with time delays cannot be inverted exactly, approximations are required.

Approximate Inverse Theorem kj 1968

Consider transfer function $P(s)$ which is proper and stable with no zeros on the imaginary axis. Let $P(s) = P^+(s)P^-(s)$ be a normalized factorization of $P(s)$ such that $P^+(s)$ has all its zeros in the left half plane and $P^-(s)$ has all its zeros in the left half plane and $P^-(0) = 1$, $P^-(s)P^-(-s) = 1$. Let the input to the system be a unit step in the reference, then the approximate inverse

$$P^\dagger(s) = \frac{1}{P^+(s)}$$

minimizes the mean square error $e = r - y$ and we have

$$\min \int_0^\infty (r(t) - y(t))^2 dt = -\frac{d \log P^-(s)}{ds}\bigg|_{s=0} = T_{ar}(P^-(s))$$

Sketch of Proof

Parsevals relation gives

$$J(H)=\int_0^\infty e^2(t)dt=\frac{1}{2\pi i}\int_{s=-i\infty}^{s=i\infty}\big(1-P(s)H(s)\big)\big(1-P(-s)H(-s)\big)\frac{ds}{-s^2}$$

The integral is finite only if $P(0)H(0)=1$, completion of squares give

$$\begin{aligned}&\big(1-P(s)H(s)\big)\big(1-P(-s)H(-s)\big)=\\&\big(1-P^+(s)H(s)\big)\big(1-P^+(-s)H(-s)\big)\\&+\big(P^+(s)-P(s)\big)H(s)+\big(P^+(-s)-P(-s)\big)H(-s)\end{aligned}$$

The function $F(s)=\big(P^+(s)-P(s)\big)H(s)s^{-2}$ is analytic in the right half plane, hence

$$\int_\Gamma F(s)ds=\int_I F(s)ds+\int_{\gamma^+}F(s)ds+\int_{\Gamma^+}F(s)ds=0$$

where I is the imaginary axis, γ^+ a small right semicircle at the origin and Γ^+ a large right semicircle.

Sketch of Proof ...

The integral along Γ^+ is zero. The integral around a small circle around the origin is obtained by residue calculus. The residue of $F(s)$ at the origin is

$$R = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{\big(P^+(s) - P(s)\big)H(s)}{s} = \lim_{s\to 0} \frac{P^+(s)\big(1 - P^-(s)\big)H(s)}{s}$$
$$= -P^+(0)P^{-'}(0)H(0) = -\frac{P^{-'}(0)}{P^-(0)} = -\frac{d\log P^-(s)}{ds}\Big|_{s=0}$$

where the last equality follows from $P(0)H(0) = 1$. Furthermore we have

$$\int_{\gamma^+} F(-s)ds = -\int_{-\gamma^+} F(-s)ds = \int_{\gamma^+} F(s)ds = \pi i R$$

Collecting the pieces we have

$$J(H) = \int_0^\infty e^2(t)dt = \frac{1}{2\pi i}\int_{s=-i\infty}^{s=i\infty} \big(1 - P(s)H(s)\big)\big(1 - P(-s)H(-s)\big)\frac{ds}{-s^2}$$
$$\geq 2R = -\frac{d\log P^-(s)}{ds}\Big|_{s=0} = T_{ar}(P^-)$$

Where equality is obtained for

$$H(s) = P^\dagger(s) = 1/P^+(s)$$

Example of Approximate Inverse

Right half plane zero

$$P(s) = \frac{a-s}{s+2}, \quad P^- = \frac{a-s}{a+s}, \quad P^+ = \frac{s+a}{s+2}, \quad P^\dagger = \frac{s+2}{s+a}$$

$$\min \int_0^\infty e^2(t)dt = -(\log P^-(s))'|_{s=0} = \frac{1}{2a}$$

slow zeros are bad

Time delay

$$P(s) = \frac{e^{-sL}}{s+1}, \quad P^- = e^{-sL}, \quad P^+(s) = \frac{1}{s+1}, \quad P^\dagger = s+1,$$

$$\int_0^\infty e^2(t)dt = \quad -(\log P^-(s))'|_{s=0} = L$$

Notice equivalence between L and $2/a$ compare $e^{-sL} \approx \dfrac{1-sL/2}{1+sL/2}$

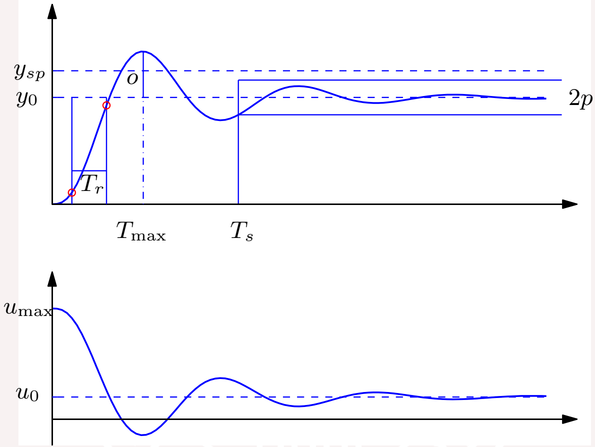
Ideal Transfer Functions $G_{yr}(s)$

Neglecting limitations of input signal amplitude and rates we have the following “ideal” transfer functions $G_{yr}(i\omega)$ obtained by feedforward compensation

- Minimum phase process: $G_{yr}(s) = 1$
- Process with right half plane zero at $s = b$: $G_{yr} = \dfrac{b-s}{b+s}$
- Process with time delay: $G_{yr} = e^{-sL}$

Notice that all transfer functions are all-pass $|G(i\omega)| = 1$

Requirements - Time domain



- Settling time T_s
- Average residence time $T_{ar} = \int_0^\infty th(t)dt$
- Rise time T_r
- Overshoot o
- Steady state error e_{ss}

Tracking Slow Signals - Error Coefficients

Tracking signals with constant velocity (ramps) or constant acceleration is some times important.

$$G_{yr}(s) = \frac{a(s)}{b(s)}, \quad G_{er}(s) = 1 - G_{yr}(s) = \frac{a(s) - b(s)}{a(s)} = \frac{\bar{b}(s)}{a(s)}$$

where

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n$$
$$b(s) = b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n$$

We have

$$G(s) = G(0) + sG'(0) + \frac{s^2}{2} G''(0) + \cdots$$

$$e(t) = e_0 r(t) + e_1 \frac{de}{dt} + e_2 \frac{d^2 e}{dt^2} + \cdots$$

$$e_0 = G(0), \quad e_1 = G'(0), \quad e_2 = G''(0)/2, \quad e_3 = G^{(3)}(0)/3!$$

Tracking Slow Signals - Error Coefficients ...

$$G(s) = G(0) + sG'(0) + \frac{s^2}{2}G''(0) + \cdots$$

$$e(t) = e_0 r(t) + e_1 \frac{de}{dt} + e_2 \frac{d^2 e}{dt^2} + \cdots$$

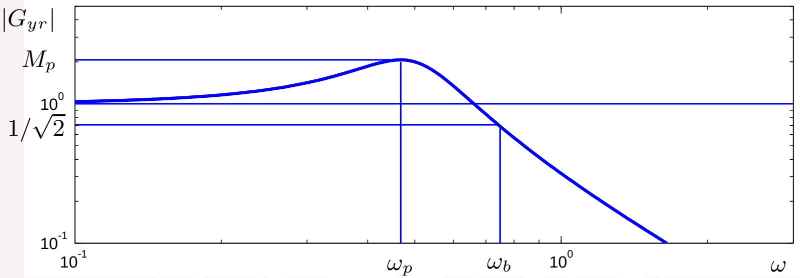
$$e_0 = G(0) \quad e_1 = G'(0) \quad e_2 = G''(0)/2 \quad e_3 = G^{(3)}(0)/3!$$

The numbers $e_k = \dfrac{G^{(k)}(0)}{k!}$, which have good physical interpretation, are called **error coefficients**

Zero tracking error for a constant input requires $G_e(0) = 0$, hence $\bar{b}_n = a_n - b_n = e_0 = 0$. If $G_{er}(0) = 0$ the steady state tracking error for a ramp input $r(t) = v_0(t)$ becomes $e_{ramp}(t) = G_{er}(0)) = e_1 v_0$.

If $G(0) = 0$ and $G'(0) = 0$ the steady state tracking errors for step and ramp inputs are zero and the steady state tracking error for an input with constant acceleration $r(t) = a_0 t^2$ is $e_{acc}(t) = e_2 a_0$

Requirements - Frequency Response



- Bandwidth ω_b
- Peak frequency ω_p frequency peak M_p

Rise Time Bandwidth Product

Consider a stable transfer function $G(s)$ and impulse response $h(t)$. Rise time is based on the unit step response several definitions:

- (1) $T_t = \dfrac{G(0)}{\max_t h(t)}$
- (2) The time for the output to go from 10% to 90% of steady state value

The bandwidth ω_{bw} can also be defined in many different ways

- (1) $\omega_{bw} G(0) = \int_0^\infty |G(i\omega)| d\omega$ (2) $|G(i\omega_{bw})| = \dfrac{\sqrt{2}}{2} G(0)$

We obtain the following estimate of the steepest slope of the unit step response

$$\max_{0 \le t \le \infty} h(t) = \max_t \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} G(i\omega) d\omega \right| \le \frac{1}{\pi} \int_0^{i\infty} |G(i\omega)| d\omega = \frac{G(0)\omega_{bw}}{\pi}$$

Definition (1) of the rise time and (2) the bandwidth give

$$T_r \omega_{bw} \le \pi$$

Examples of Bandwidth-Risetime Product

Here we use $|G(i\omega_{bw})| = G(0)\sqrt{2}/2$

First order system

$$P(s) = \frac{a}{s+a} \quad T_r = \frac{1}{a}, \quad \omega_{bw} = a, \quad T_r\,\omega_{bw} = 1$$

Second order system

$$P(s) = \frac{a^2}{(s+a)^2} \quad T_r = \frac{e}{a} = \frac{2.72}{a}, \quad \omega_{bw} = a\sqrt{\sqrt{2}-1} \quad T_r\,\omega_{bw} = 1.75$$

Second order system $P(s) = \dfrac{1}{s^2 + 2\zeta s + 1}$,

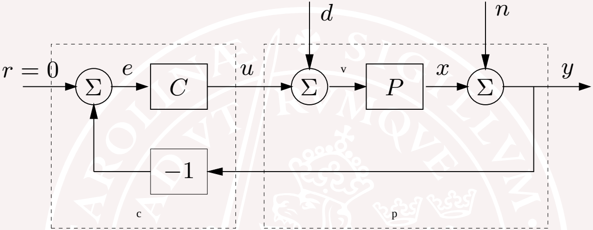
$$T_r\,\omega_{bw} = e^{\phi/tan\phi}\sqrt{1 - 2\zeta^2 + \sqrt{(1-2\zeta^2)^2 + 1}}, \quad \phi = \arccos\zeta$$

- $\zeta = 0.707$ gives $T_r\omega_{bw} = 2.19$
- $\zeta = 1$ gives $T_r \times \omega_{bw} = 1.75$.
- A reasonable rule of thumb is $T_r\,\omega_{bw} \approx 2(1-3)$

Requirements

Theme: Requirements for design and verification

Effect of Feedback on Disturbances



Output without control $Y_{ol} = N + PD$

Output with feedback control

$$Y_{cl} = \frac{1}{1+PC}(N+PD) = SY_{ol}$$

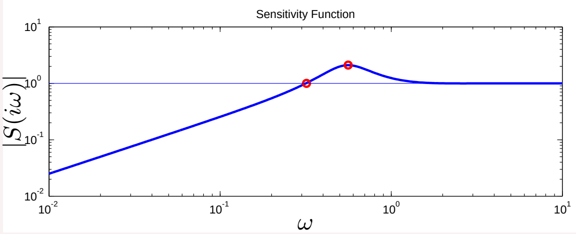
The sensitivity function $S = 1/(1+PC)$ tells how feedback influences the effect of disturbances. Disturbances with frequencies such that $|S(i\omega)| < 1$ are reduced by feedback, disturbances with frequencies such that $|S(i\omega)| > 1$ are amplified by feedback.

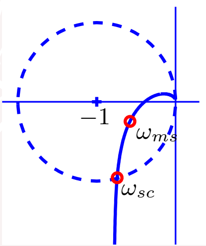
Assessment of Disturbance Reduction

We have

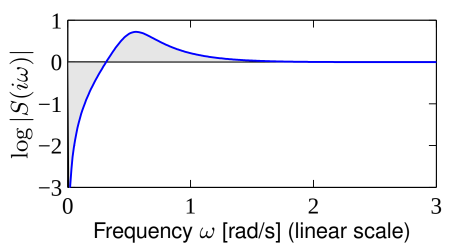
$$Y_{cl} = SY_{ol}(t), \qquad S(s) = \frac{1}{1 + P(s)C(s)}$$

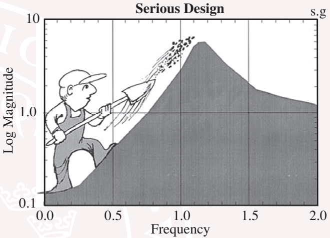
- Feedback attenuates disturbances when $|S(i\omega)| < 1$
- Feedback amplifies disturbances when $|S(i\omega)| > 1$
- The sensitivity crossover frequency ω_{sc} ($S(i\omega_{sc}) = 1$) is an important parameter, (there may be many values)





The Water Bed Effect - Bode's Integral





$$\int_0^\infty \log|S(i\omega)|d\omega = \pi \sum \mathrm{Re}\, p_k - \frac{\pi}{2}K_v, \quad K_v = \lim_{s\to\infty} sL(s)$$

p_i RHP pole. The sensitivity can be decreased at one frequency at the cost of increasing it at another frequency. Feedback design is a trade-off!.

$$\int_0^\infty \omega^2 log|T(i\omega)|d\omega = \pi\tau + \pi \sum \mathrm{Re}\frac{1}{z_i} - \frac{\pi}{2K_v}$$

The First IEEE Bode Lecture 1989

FEATURE

Respect the Unstable

The practical, physical (and sometimes dangerous) consequences of control must be respected, and the underlying principles must be clearly and well taught.

By Gunter Stein

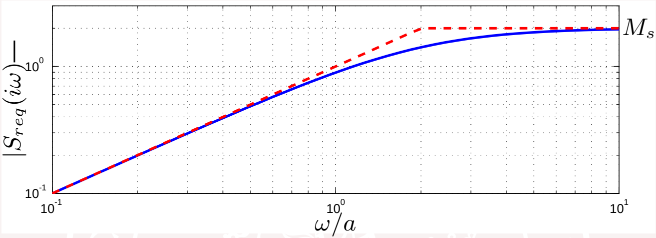
Gunter Stein's Bode Lecture

A video was made by IEEE and the Lecture was published in the IEEE Control Systems Magazine in 2003!

http://www.ieeecss-oll.org/lectures/1989/respect-unstable


IEEE
Control Systems
Magazine
August 2003
Volume 23 Number 4
Respect
the Unstable

Right Half Plane Zeros



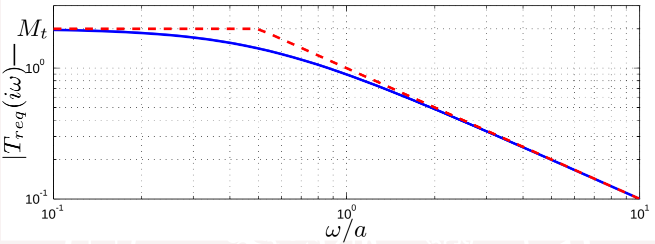
Consider a process with a real right half plane zero at $s = z$, assume that we want the sensitivity function S to be below a given specification S_{req} where

$$S_{req}(s) = \frac{sM_s}{s + aM_s}, \quad |S(i\omega)| < |S_{req}(i\omega)|, \quad \left|\frac{S(i\omega)}{S_{req}(i\omega)}\right| < 1$$

Since $S(z) = 1$ and S_{req} is regular and analytic in the RHP it follows from the maximum modulus theorem that a conservative requirement is that $S_{req}(z) > 1$. Hence

$$zM_s > z + aM_s \quad \Rightarrow \quad a < z\frac{M_s - 1}{M_s}$$

Right Half Plane Poles



Consider a process with a real right half plane pole at $s = p$, assume that we want the sensitivity function T to be below a given specification T_{req} where

$$T_{req}(s) = \frac{a}{s + a/M_t}, \quad |T(i\omega)| < |T_{req}(i\omega)|, \quad \left|\frac{T(i\omega)}{T_{req}(i\omega)}\right| < 1$$

Since $T(p) = 1$ and T_{req} is regular and analytic in the RHP it follows from the maximum modulus theorem that a conservative requirement is that $T_{req}(p) > 1$. Hence

$$a > p + a/M_t \quad \Rightarrow \quad a > p\frac{M_t}{M_t - 1}$$

Time Delays and RHP Zeros Impose Limitations

A RHP zero $s = z$ limit the sensitivity crossover frequency ω_{bw}

$$\omega_{bw} \approx< z\frac{M_s - 1}{M_s}$$

A time delay L limits (based on $e^{-sL} \approx \dfrac{1 - sL/2}{1 + sL/2}$

$$\omega_{tc} \approx> \frac{2}{L}$$

A RHP pole $s = p$ requires a high sensitivity crossover frequency ω_{bw}

$$\omega_{bw} > p\frac{M_t}{M_t - 1}$$

Poles and zeros in the RHP give high sensitivity if they are too close

$$M_s \geq \left|\frac{p + z}{p - z}\right|$$

A right half plane pole p and a time delay L limit the sensitivity

$$M_t \geq e^{pL}$$

RHP Pole and Zero

Consider a process $P(s)$ with a pole p and a zero z in the RHP

$$P(s) = \frac{s-z}{s-p}\bar{P}(s)$$

The sensitivity function is

$$S(s) = \frac{1}{1+P(s)C(s)} = \frac{s-p}{s-p+(s-z)\bar{P}(s)C(s)} \qquad S(z) = 1$$

Introduce the weight

$$w_p(s) = \frac{s+p}{s-p}, \qquad |w_p(i\omega)| = 1$$

The function $w_p(s)S(s)$ is then regular in the RHP and the maximum modulus theorem gives

$$M_s = \max_\omega |S(i\omega)| = \max_\omega |w_p(i\omega)S(i\omega)| \geq |w_p(z)S(z)| = \left|\frac{z+p}{z-p}\right|$$

RHP Pole and Time Delay

Consider a process with a pole p in the RHP and a time delay L

$$P(s) = \frac{e^{-sL}}{s-p}\bar{P}(s)$$

The complementary sensitivity function is

$$T(s) = \frac{P(s)C(s)}{1+P(s)C(s)} = \frac{e^{-sL}\bar{P}(s)C(s)}{s-p+e^{-sL}\bar{P}(s)C(s)}, \quad T(p) = 1$$

Introduce the weight

$$w_L(s) = e^{sL}, \qquad |w_L(i\omega)| = 1$$

The function $w_L(s)T(s)$ is then regular in the RHP and the maximum modulus theorem gives

$$M_t = \max_\omega |T(i\omega)| = \max_\omega |w_L(i\omega)T(i\omega)| \geq |w_L(p))T(p)| = e^{pL}$$

Complex RHP Pole and Zero

RHP zero at $s = x_0 \pm iy_0$

$$\omega_{sc} < \frac{\sqrt{M_s^2 x_0^2 + (M_s^2 - 1)y_0^2} - x_0}{M_s}$$

Pure real or imaginary zeros

$$\omega_{sc} < x_0 \frac{M_s - 1}{M_s}, \qquad \omega_{sc} < \frac{\sqrt{M_s^2 - 1}}{M_s} y_0$$

RHP pole at $s = x_0 \pm iy_0$

$$\omega_{st} > \frac{x_0 M_1 + \sqrt{M_t^4 x_0^2 + M_t^2 (M_t^2 - 1) y_0^2} + M_t x_0}{\sqrt{M_t^2 - 1}}$$

Pure real or imaginary poles

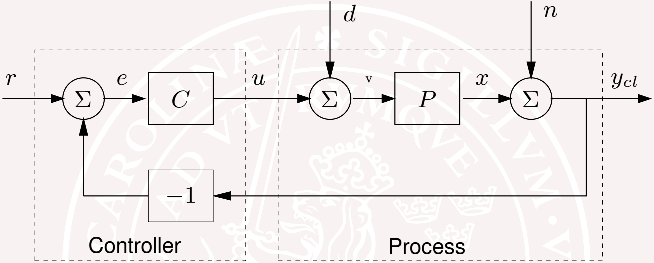
$$\omega_{st} > \frac{M_t}{M_t - 1} x_0, \qquad \omega_{st} > \frac{M_t}{\sqrt{M_t^2 - 1}} y_0$$

Tune for Load Disturbances

G. Shinskey Intech Letters 1993: *"The user should not test the loop using set-point changes if the set point is to remain constant most of the time. To tune for fast recovery from load changes, a load disturbance should be simulated by stepping the controller output in manual, and then transferring to auto. For lag-dominant processes, the two responses are markedly different."*

Motion control is different!

Load Disturbance Attenuation



Load disturbances typically have low frequencies, low frequency approximations are relevant. Transfer functions

$$G_{yd} = \frac{P}{1+PC} = PS = \frac{T}{C} \approx \frac{1}{C} \qquad G_{ud} = \frac{PC}{1+PC} = T \approx 1$$

For controller with integral action we have for small s

$$G_{yd} \approx s/k_i$$

Load Disturbance Attenuation

Low frequencies

$$\frac{P}{1+PC} \approx \frac{1}{C} \approx \frac{s}{k_i}$$

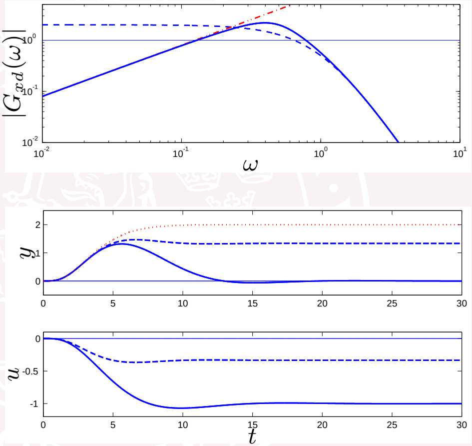
High frequencies

$$\frac{P}{1+PC} \approx P$$

Maximum error

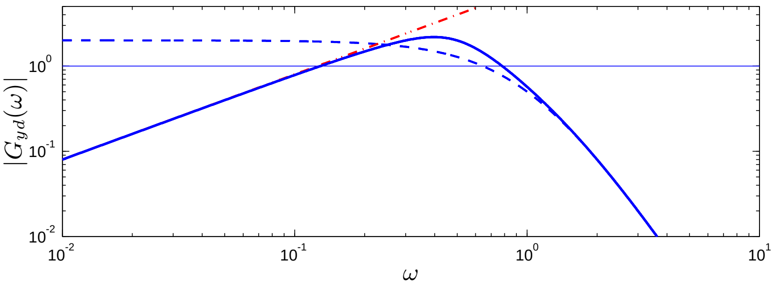
$$e_{max} \approx \frac{P(0)}{1+k_pP(0)}$$

$P = 2/(s+1)^4$ PI: $k_p = 0.25$, $k_i = 0.125$



P dashed, PI full

Frequency Domain Criteria



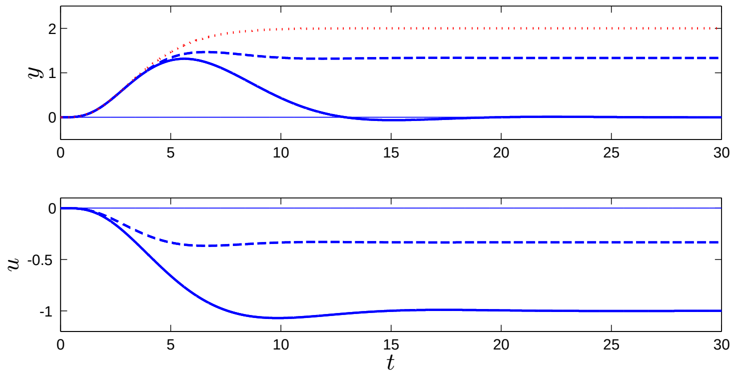
Approximations for different frequencies

$$G_{yd} \approx \frac{s}{k_i}, \qquad |G_{yd}| \leq M_s|P|, \qquad G_{yd} = \frac{P}{1+PC} = PS \approx P$$

Criteria: $\omega_s \approx K\, k_i \quad \max|G_{yd}(i\omega)| \approx KM_s \quad K = P(0)$

Time Domain Criteria

Error and control signals for unit step load disturbance at process input



Criteria: e_{max} maximum error, t_{max} time when maximum occurs, $IAE \approx IE = 1/k_i$

Time Domain Criteria

Calculated control error for unit step disturbance.

Peak error

$$e_{\max} = \max_{0 \le t < \infty} |e(t)|, \qquad t_{\max} = \arg\max |e(t)|.$$

Integral criteria

$$IE = \int_0^\infty e(t)dt, \quad IAE = \int_0^\infty |e(t)|dt, \quad ISE = \int_0^\infty e(t)^2 dt$$

$$QE = \int_0^\infty (e^2(t) + \rho u^2(t))dt,$$

Integral Error IE and Integral Gain k_i

PID control or any controller with integral action

$$u(t) = ke(t) + k_i \int_0^t e(t)dt - k_d \frac{dy}{dt}.$$

Assume that the closed-loop system is stable. Apply a unit step load disturbance at process input.

$$u(\infty) - u(0) = k_i \int_0^\infty e(t)dt.$$

The change in control signal is equal to the change of the disturbance, $u(\infty) - u(0) = 1$, hence

$$IE = \int_0^\infty e(t)dt = \frac{1}{k_i}.$$

Integral gain k_i is thus inversely proportional to the integrated error caused by a unit step load disturbance applied to the process input.

Approximations for PID Control

Response of y to load disturbance d is characterized by

$$\frac{P}{1+PC} \approx \frac{s}{k_i}, \quad M_s, \quad \omega_s$$

Response of u to measurement noise n is characterized by

$$\frac{C}{1+PC} = SC \approx \frac{s}{1+Kk_i}\,\frac{k_i + k_p s + k_d s^2}{s(1+sT_f+(sT_f)^2)}, \quad K = P(0)$$

Robustness to process variations is characterized by

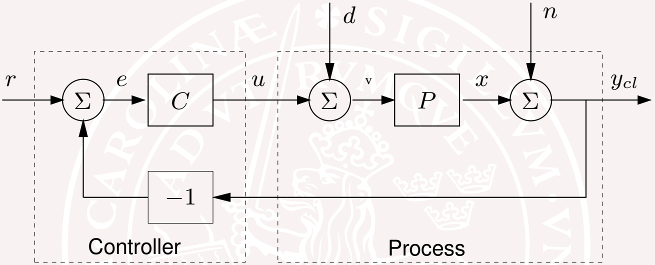
$$S = \frac{1}{1+PC}, \quad T = \frac{PC}{1+PC}, \quad M_s, \quad M_t$$

Responses of y and u to reference signal r is characterized by

$$\frac{PCF}{1+PC}, \quad \frac{CF}{1+PC}$$

Use setpoint weighting or feedforward

Measurement Noise Injection

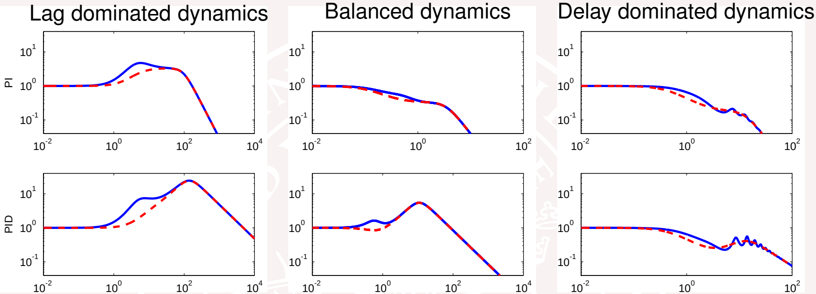


Measurement noise typically has high requencies (why). High frequency approximations are relevant. Transfer functions

$$G_{yn} = \frac{1}{1+PC} = S \approx 1 \qquad -G_{un} = \frac{C}{1+PC} = CS \approx C$$

High frequency roll-off of C is very important

Approximations of G_{un} for PID Control



Blue lines true transfer function red lines approximations

$$-G_{un}(s) = \frac{C}{1+PC} = -SC$$

$$\approx \frac{k_i + k_p s + k_d s^2}{(s+Kk_i)(1+sT_f+(sT_f)^2/2)}$$

$$K = P(0) \neq \infty$$

More details in PID lecture

Requirements for Noise Injection

- Difficult to obtain good information about noise spectrum
- Control actions generated by the noise is the important factor
- Captured by $G_{un} = \frac{C}{1+PC} = SC \approx C$
- High frequency roll-off of C (noise filter) is important
- $\max|G_{un}(i\omega)|, ||G_{un}||^2 = \int_{-\infty}^{\infty} |G_{un}(i\omega)|^2 d\omega$

Requirements

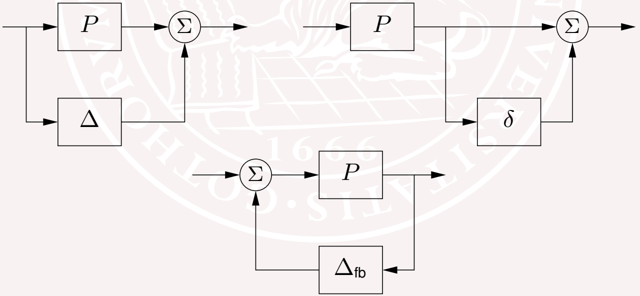
Theme: Requirements for design and verification

Process Uncertainty

- Process dynamics may change
- Feedback can deal with process variations
- How to characterize uncertainty

  Parameter variations, more general variations, unmodeled dynamics

- Main results (the usual suspects S and T)
- Additive Δ, multiplicative δ and feedback uncertainty Δ_{fb}



When are Two Systems Close

- For stable systems

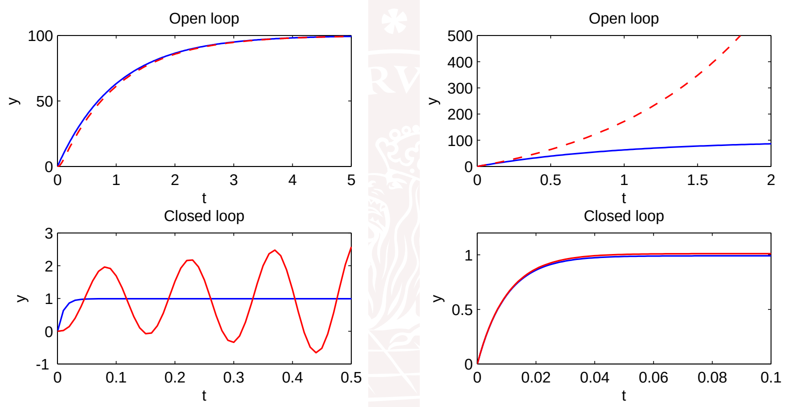
$$\delta(P_1, P_2) = \max_{\omega} |P_1(i\omega) - P_2(i\omega)|$$

as a measure of of closeness of two processes.

- Is this a good measure?
- Are there other alternatives?
- A long story

Gap metric (Zames)
Graph metric coprime factorization (Vidyasagar) $G = N/D$
Vinnicombe's metric

When are Two Systems Close?

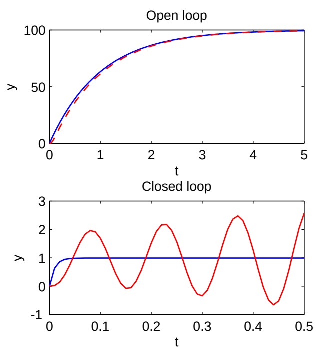


Comparing step responses can be misleading!

Frequency responses are better

Better to compare closed loop responses

Similar Open Loop Different Closed Loop



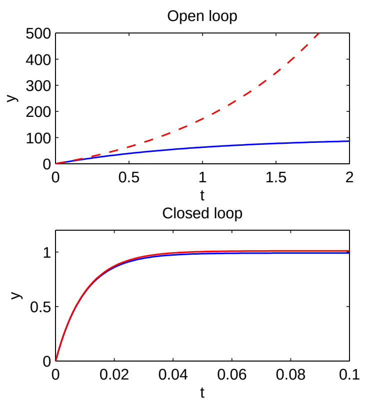
$$P_1(s) = \frac{100}{s+1}, \qquad P_2(s) = \frac{100}{(s+1)(1+0.025s)^2}$$

Complementary sensitivity functions with unit feedback $C = 1$

$$T_1 = \frac{100}{s+101}, \qquad T_2 = \frac{1.616e5}{(s+83.9)(s^2-2.90s+1926s+1926)}$$

Very different closed loop systems

Different Open Loop Similar Closed Loop



Systems and complementary sensitivity functions

$$P_1(s) = \frac{100}{s+1}, \quad T_1(s) = \frac{100}{s+101}, \quad P_2(s) = \frac{100}{s-1}, \quad T_2(s) = \frac{100}{s+99}$$

Closed loop systems are very similar

The Graph Metric

We know how to compare stable systems. What to do with unstable systems? Let

$$P(s) = \frac{B(s)}{A(s)}$$

where A and B are polynomials. Choose a stable polynomial C whose degree is not lower than the degrees of A and B, then

$$P(s) = \frac{\dfrac{B(s)}{C(s)}}{\dfrac{A(s)}{C(s)}} = \frac{N(s)}{D(s)}$$

Compare the numerator and denominator transfer functions jointly.

How to Choose D and N

Two rational functions D and N are called coprime if there exist rational functions X and Y which satisfy the equation

$$XD + YN = 1$$

The condition for coprimeness is essentially that $D(s)$ and $N(s)$ do not have any common factors.

Let $D^*(s) = D(-s)$. A factorization $P = N/D$ such that

$$DD^* + NN^* = 1$$

is called a normalized coprime factorization of P.

Robustness

Additive perturbations $P \to P + \Delta P$, ΔP stable

$$\frac{|\Delta P(i\omega)|}{|P(i\omega)|} < \frac{|P(i\omega)C(i\omega)|}{|1+P(i\omega)C(i\omega)|} = \frac{1}{|T(i\omega)|}$$

For normalized Co-prime factor perturbations
$P = N/D \ \to \ (N+\Delta N)(D+\Delta D)$ this generalizes to

$$||(\Delta N(i\omega), \Delta D(i\omega))|| < \frac{1}{\gamma(\omega)}$$

where $s_m = \dfrac{1}{\max_\omega \gamma(\omega)}$ is a generalized ($s_m = 1/M_s$) stability margin

$$\gamma = \bar{\sigma}\begin{pmatrix} \dfrac{1}{1+P(i\omega)C(i\omega)} & \dfrac{P(i\omega)}{1+P(i\omega)C(i\omega)} \\ \dfrac{P(i\omega)}{1+P(i\omega)C(i\omega)} & \dfrac{P(i\omega)C(i\omega)}{1+P(i\omega)C(i\omega)} \end{pmatrix}$$
$$= \frac{\sqrt{(1+|P(i\omega)|^2)(1+|C(i\omega)|^2)}}{|1+P(i\omega)C(i\omega)|}$$

Vinnicombe's Metric

Consider two systems with the normalized coprime factorizations

$$P_1 = \frac{D_1}{N_1}, \qquad P_2 = \frac{D_2}{N_2}$$

To compare the systems it must be required that

$$\frac{1}{2\pi}\Delta \arg_\Gamma(N_1 N_2^* + D_1 D_2^*) = 0$$

where Γ is the Nyquist contour. In the polynomial representation this condition implies

$$\frac{1}{2\pi}\Delta \arg_\Gamma(B_1 B_2^* + A_1 A_2^*) = \deg A_2$$

The winding number constraint!

Vinnicombe's Metric

If the winding number constraint is satisfied Vinnicombe's Metric can be defined as

$$\delta_\nu(P_1, P_2) = \sup_\omega \frac{|P_1(i\omega) - P_2(i\omega)|}{\sqrt{(1 + |P_1(i\omega)|^2)(1 + |P_2(i\omega)|^2)}}$$

Examples:

$$P_1(s) = \frac{100}{s+1}, \quad P_2(s) = \frac{100}{(s+1)(1+0.025s)^2}, \quad \delta(P_1, P_2) = 0.98$$

$$P_1(s) = \frac{100}{s+1}, \quad P_2(s) = \frac{100}{s-1}, \quad \delta(P_1, P_2) = 0.02$$

Feedback Interpretation

Consider systems with the transfer functions P_1 and P_2. Compare the complementary sensitivity functions for the closed loop systems obtained with a controller C that stabilizes both systems.

$$\delta(P_1, P_2) = \Big| \frac{P_1 C}{1 + P_1 C} - \frac{P_2 C}{1 + P_2 C} \Big| = \Big| \frac{(P_1 - P_2) C}{(1 + P_1 C)(1 + P_2 C)} \Big|$$

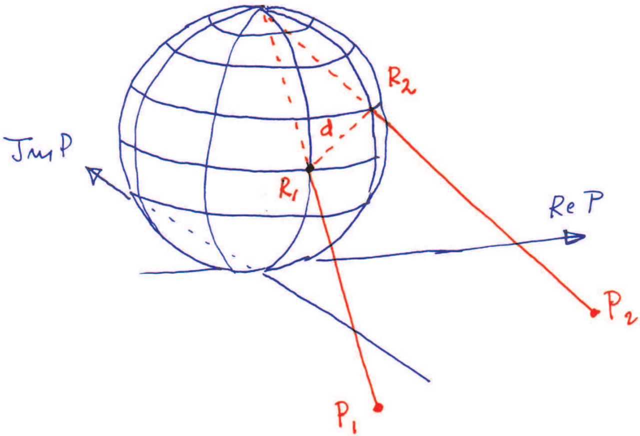
We have

$$\delta(P_1, P_2) \le M_{s1} M_{s2} |C(P_1 - P_2)|$$

It can be shown that δ is a good measure of closeness of processes. More in lecture about $\mathcal{H}_\infty$

Vinnicombes metric corresponds to $C = 1$, i.e. unit feedback.

Geometric Interpretation



Requirements

1. Introduction
2. The basic feedback system
3. A broad view of control system design
4. Command signal following - System inversion
5. Disturbances
6. The sensitivity functions - Fundamental limitations
7. Process uncertainty
8. Robustness
9. Summary

Theme: Requirements for design and verification

Small Process Variations

$$T = \frac{PC}{1+PC}, \qquad \frac{dT}{T} = \frac{1}{1+PC}\,\frac{dP}{P} = S\,\frac{dP}{P}$$

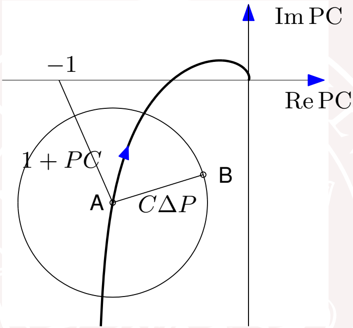
$$S = \frac{1}{1+PC}, \qquad \frac{dS}{S} = \frac{-PC}{1+PC}\,\frac{dP}{P} = -T\,\frac{dP}{P}$$

$$G_{yd} = \frac{P}{1+PC}, \qquad \frac{dG_{yd}}{G_{yd}} = S\,\frac{dP}{P}$$

$$G_{un} = \frac{C}{1+PC}, \qquad \frac{dG_{un}}{G_{un}} = -T\,\frac{dP}{P}$$

- Recall properties of S and T
- $S + T = 1$
- S small at low frequencies $S \approx 1$ at high frequencies
- T small at high frequencies $T \approx 1$ at low frequencies

Large Process Variations

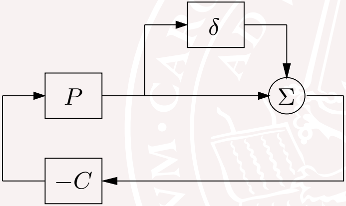


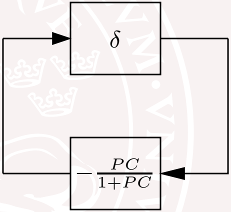
$$|C\Delta P| < |1 + PC|, \qquad \left|\frac{\Delta P}{P}\right| < \left|\frac{1+PC}{PC}\right| = \frac{1}{|T|}$$

- Large variations permitted when T is small
- Small variations when T is large, $M_t = \max |T(i\omega)|$

Another View of Robustness

A feedback system where the process has multiplicative uncertainty, i.e. $P(1+\delta)$, where δ is the relative error, can be represented with the following block diagrams





The small gain theorem gives the stability condition

$$|\delta| < \left|\frac{1+PC}{PC}\right| = \frac{1}{|T|}$$

Same result as obtained before!

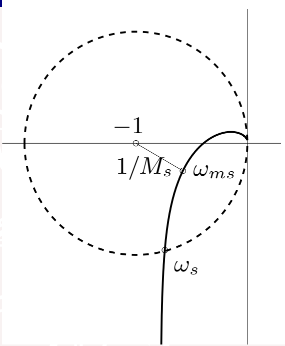
Robustness and Sensitivity

Gain margin

$$g_m \geq \frac{M_s}{M_s - 1}$$

Phase margin

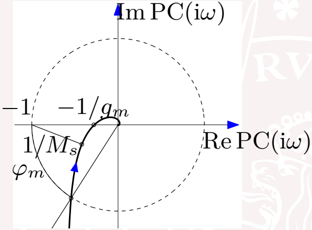
$$\varphi_m \geq 2 \arcsin \frac{1}{2M_s}$$

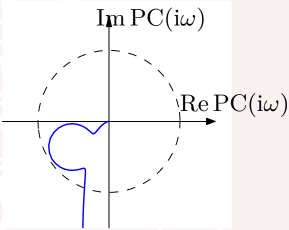


Constraints on both gain and phase margins can be replaced by constraints on M_s.

- $M_s = 2$ guarantees $g_m \geq 2$ and $\varphi_m \geq 30°$
- $M_s = 1.6$ guarantees $g_m \geq 2.7$ and $\varphi_m \geq 36°$
- $M_s = 1.4$ guarantees $g_m \geq 3.5$ and $\varphi_m \geq 42°$
- $M_s = 1$ guarantees $g_m = \infty$ and $\varphi_m \geq 60°$

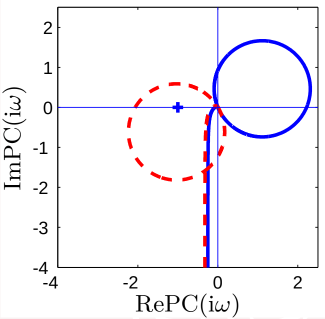
Stability Margins

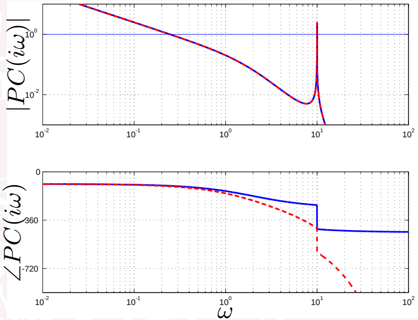




- Necessary to specify both g_m and φ_m
- Not sufficient to specify both g_m and φ_m (right figure)
- M_s can replace φ_m and g_m
- φ_m and g_m are widely used in industry difficulties when the Nyquist curve has warts

Delay Margin





Time delay required to make the system unstable

Peaks in the loop transfer function PC are dangerous

They are often caused by resonances

$$P(s) = \frac{100}{s(0.5s+1)^2(s^2+0.004s+100)}, \quad C(s) = \frac{0.25}{s}, \quad L = 0.3$$

Requirements

1. Introduction
2. The basic feedback system
3. A broad view of control system design
4. Command signal following - System inversion
5. Disturbances
6. The sensitivity functions - Fundamental limitations
7. Process uncertainty
8. Robustness
9. Summary

Theme: Requirements for design and verification

Summary

- Necessary with a broad view: large and small signals, dynamics

  Limits on control signals and their rate
- Fundamental limitations: time delays and RHP poles and zeros

  Consider adding or shifting sensors
- Requirements: performance and robustness

  Load disturbance attenuation
  Measurement noise injection
  Command signal following
  Robustness
- The closed loop system is characterized by a collection of transfer functions:

  Error feedback Gang of Four: S, T, PC, CS
  System with 2DOF: FS, TF, CSF
- Requirements are based on properties of these transfer functions

  Bode plots, time responses or their features

Properties of the Sensitivity Function

- Can the sensitivity be small for all frequencies?

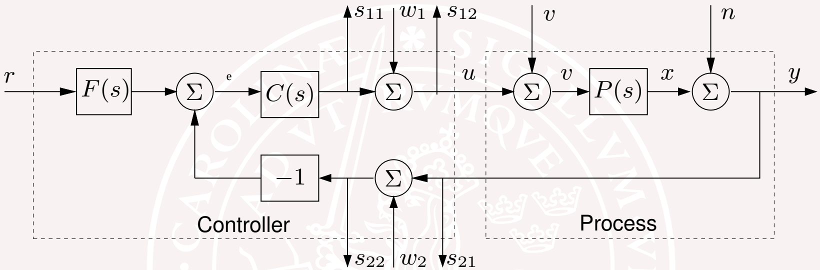
  No we have $S(\infty) = 1$!
- Can we have $|S(i\omega)| \leq 1$?

  If the Nyquist curve of $L = PC$ is in the first and fourth quadrant! Passive systems!
- Bode's integral, p_k RHP poles of $L(s)$

$$\int_0^\infty \log|S(i\omega)| d\omega = \pi \sum \operatorname{Re} p_k - \frac{\pi}{2} \lim_{s\to\infty} sL(s)$$

- The "water-bed effect". Push the curve down at one frequency and it pops up at another!
- Time delays, poles and zeros in the RHP limit performance
- Useful to let the loop transfer function go to zero rapidly for high frequencies (*high-frequency roll-off*)

Testing Requirements



- S Change w_1 measure s_{12} or change w_2 measure s_{22}
- T Change w_1 measure s_{11} or change w_2 measure s_{21}
- G_{yv} Change w_1 measure s_{21}
- G_{un} Change w_2 measure s_{11}
- TF Change y_{sp} measure s_{21}
- SFC Change y_{sp} measure s_{11}

Transfer functions and Parameters

Requirement	Transfer functions	Parameters
General assessment	P, P_{nmp}, PC, S	$\omega_{gc}, \omega_{sc}, n_{gc}, T_{ar}$
Robustness	S, T, PC	$M_s, M_t, g_m, \varphi_m, \delta_m$
Load disturbances	S, G_{yd}, G_{ud}	M_s, IE, IAE, e_{max}, t_{emax}
Measurement noise	G_{un}, G_{yn}	$\|\|G_{un}\|\|_\infty, \|\|G_{un}\|\|_2$
Command signal	$G_{yr},\ G_{ur}, T$	$T_{rise}, T_{settling}, o, \omega_B, e_v$