Pole Placement Design

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Pole Placement Design

1. Introduction
2. Simple Examples
3. Polynomial Design
4. State Space Design
5. Robustness and Design Rules
6. Model Reduction
7. Oscillatory Systems
8. Summary

Theme: Be aware where you place them!

Introduction

- A simple idea
- Strong impact on development of control theory
- The only constraint is reachability and observability
- The robustness debate

  Classic control vs State feedback
- Easy to apply for simple systems
- Polynomial equations notoriously badly conditioned: $z^n = 0$
  - OK for low order systems, use matrix formulation for high order systems
- How to choose closed loop poles - The Million $ question

  How do the closed loop poles influence performance
  How do the closed loop poles influence robustness
  A bit of history - Mats Lilja's PhD thesis TFRT 1031 (1989)
  Insight into model reduction

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Control of First Order Systems

- State: variables required to characterize storage of mass, momentum and energy
- Many systems are approximately of first order
- The key is that the storage of mass, momentum and energy can be captured by one parameter
- Examples
  - Velocity of car on the road
  - Control of velocity of rotating system
  - Electric systems where energy storage is essentially in one capacitor or one inductor
  - Incompressible fluid flow in a pipe
  - Level control of a tank
  - Pressure control in gas tank
  - Temperature in a body with essentially uniform temperature distribution (e.g. steam filled vessel)

PI Control of First Order Systems

Process, controller and loop- transfer function

$$P(s) = \frac{b}{s+a}, \qquad C_{fb}(s) = k_p + \frac{k_i}{s}, \qquad C_{ff} = \beta k_p + \frac{k_i}{s}$$

Closed loop transfer functions with error feedback

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{b(k_p s + k_i)}{s^2 + (a + bk_p)s + bk_i}$$

$$G_{yd}(s) = \frac{P}{1+PC} = \frac{bs}{s^2 + (a + bk_p)s + bk_i}$$

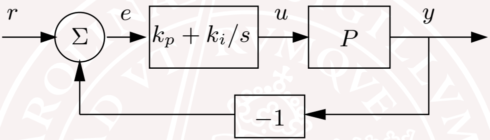
Controller with set point weighting

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{b(\beta k_p s + k_i)}{s^2 + (a + bk_p)s + bk_i}$$

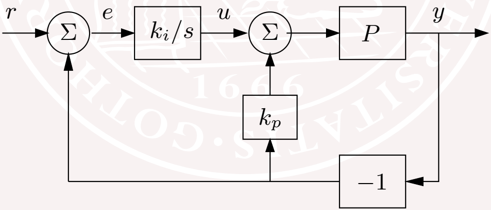
Poles chosen by controller gains k_p, k_i, zero by set-point weight β

Block Diagram Representations

Error feedback. Controller gives the zero $-k_i/k_p$ in $G_{yr}(s)$



Controller with two degrees of freedom with $\beta = 0$. Controller generates no zeros in $G_{yr}(s)$.



Gang of Seven

Transfer function of Gang of Four

$$S = \frac{s(s+a)}{s^2 + (a + bk_p)s + bk_i}, \qquad PS = \frac{bs}{s^2 + (a + bk_p)s + bk_i}$$

$$CS = \frac{(k_p s + k_i)(s+a)}{s^2 + (a + bk_p)s + bk_i}, \qquad T = \frac{b(k_p s + k_i)}{s^2 + (a + bk_p)s + bk_i}$$

Transfer functions from command signal

$$G_{yr}(s) = \frac{b(\beta k_p s + k_i)}{s^2 + (a + bk_p)s + bk_i}$$

$$G_{ur}(s) = \frac{(\beta k_p s + k_i)(s+a)}{s^2 + (a + bk_p)s + bk_i}$$

$$G_{er}(s) = \frac{s(s + a + (1-\beta)bk_p)}{s^2 + (a + bk_p)s + bk_i}$$

Second Order Systems

- Two states because storage of mass, momentum and energy can be captured by two parameter
- Examples
  - Position of car on the road
  - Control of angle of rotating system
  - Stabilization of satellites
  - Electric systems where energy is stored in two elements (inductors or capacitors)
  - Levels in two connected tanks
  - Pressure in two connected vessels

PD Control of Second Order System

Process and controller transfer functions

$$P(s) = \frac{b}{s^2 + a_1 s + a_2}, \qquad C(s) = k_p + k_d s$$

Closed loop transfer function from reference to output

$$G_{yr}(s) = \frac{PC}{1 + PC} = \frac{b(k_d s + k_p)}{s^2 + (a_1 + bk_d)s + a_2 + bk_p}$$

Closed loop system of second order, controller has two parameters. All closed loop poles can be chosen, but no integral action. With setpoint weighting

$$G_{yr}(s) = \frac{PC}{1 + PC} = \frac{b(\gamma k_d s + k_p)}{s^2 + (a_1 + bk_d)s + a_2 + bk_p}$$

PI Control of Second Order Systems

Process and controller transfer functions

$$P(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}, \qquad C(s) = k_p + \frac{k_i}{s}$$

Closed loop transfer function from reference r to output y

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{(k_p s + k_i)(b_1 s + b_2)}{s^3 + (a_1 + b_1 k_p)s^2 + (a_2 + b_1 k_i + b_2 k_p)s + b_2 k_i}$$

Closed loop system of third order, controller has only two parameters. Not enough degrees of freedom. A more complex controller is required to choose closed loop characteristic polynomial.

PID Control of Second Order System

Process and controller transfer functions

$$P(s) = \frac{b_1 s + b_2}{s^2 + a_1 s + a_2}, \qquad C(s) = k_p + \frac{k_i}{s} + k_d s$$

Closed loop transfer function from reference r to output y

$$G_{yr}(s) = \frac{(b_1 s + b_2)(k_d s^2 + k_p s + k_i)}{(1 + b_1 k_d)s^3 + (a_1 + b_1 k_p + b_2 k_d)s^2 + (a_2 + b_1 k_i + b_2 k_p)s + b_2 k_i}$$

All closed loop poles can be chosen arbitrarily; with setpoint weighting

$$G_{yr}(s) = \frac{(b_1 s + b_2)(\gamma k_d s^2 + \beta k_p s + k_i)}{(1 + b_1 k_d)s^3 + (a_1 + b_1 k_p + b_2 k_d)s^2 + (a_2 + b_1 k_i + b_2 k_p)s + b_2 k_i}$$

some zeros of G_{yr} can also be chosen arbitrarily weighting, but process zero remains.

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Polynomial Design

Process and controller

$$d_p(s)Y(s) = n_p(s)U(s), \qquad d_c(s)U(s) = n_{cff}(s)R(s) - n_c(s)Y(s)$$

Closed loop transfer function

$$G_{yr}(s) = \frac{n_p(s)n_{cff}(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)} = \frac{n_p(s)n_{cff}(s)}{d_{cl}(s)}$$

Determine $d_c(s)$ and $n_c(s)$ to give the desired closed loop polynomial $d_{cl}(s)$. The zeros can be partially influenced through $n_{cff}(s)$.

Parameter count

- $\deg d_c + \deg n_c + 1 = \deg d_{cl}$
- Introduce unknown coefficients and solve linear equation

The Diophantine Equation

The equation

$$3x + 2y = 1,$$

where x and y are integers, has the solution: $x = 1$ and $y = -1$. Many other solutions can be obtained by adding 2 to x and subtracting 3 from y.

The equation

$$6x + 4y = 1,$$

cannot have a solution, because the left hand side is even and the right hand side is odd.

The equation

$$6x + 4y = 2,$$

has a solution, because we can divide by 2 and obtain the first equation.

Main Result

Let a, b, c, x and y be integers, the equation

$$ax + by = c$$

has a solution if and only if the greatest common factor of a and b divides c. If the equation has a solution x_0 and y_0 then $x = x_0 - bn$ and $y = y_0 + an$, where n is an arbitrary integer, is also a solution.

- Integers and polynomials same algebra, add, subtract, divide with remainder (size replaced by degree)
- Euclid's algorithm holds for polynomials (the same algebra add & mult!)

Proof - Euclid's Algorithm

Assume that the degree of a is greater than or equal to the degree of b. Let $a^0 = a$ and $b^0 = b$. Iterate the equations

$$a^{n+1} = b^n, \qquad b^{n+1} = a^n \mod b^n$$

until $b^{n+1} = 0$. The greatest common divisor is then $g = b^n$. If a and b are co-prime we have $b^n = 1$. Back-tracking we find that

$$ax + by = b^n = g$$

where the polynomials x and y can be found by keeping track of the quotients and the remainders in the iterations. When a and b are co-prime ($g = 1$) we get

$$ax + by = 1$$

Multiplying x and y by c gives the original equation $ax + by = c$. When a and b have a common factor the largest common divisor of a and b must be a factor of c.

An Algorithm

Let g be the greatest common divisor of a and b and let u and v be the minimal degree solutions to $ax + by = 0$.

$$\begin{pmatrix} x & y \\ u & v \end{pmatrix} \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} = \begin{pmatrix} g & x & y \\ 0 & u & v \end{pmatrix}$$

Make row transformations to transform (Gaussian elimination)

$$A^{(0)} = \begin{pmatrix} a & 1 & 0 \\ b & 0 & 1 \end{pmatrix} \quad \text{to} \quad A^{(n)} = \begin{pmatrix} g & x & y \\ 0 & u & v \end{pmatrix}$$

It follows from Euclid's algorithm that $g = A_{11}^{(n)}$ is the largest common divisor of a and b, and that a and b are co-prime if and only if $A_{11}^{(n)} = 1$. The equation

$$ax + by = c$$

has a solution if $A_{11}^{(n)}$ is a factor of c.

Non-uniqueness I

Closed loop characteristic polynomial

$$d_p d_c + n_p n_c = d_{cl}, \qquad C = n_c/d_c$$

If d_{c0}, n_{c0} is a solution then so is $d_{c0} - qn_p, n_{c0} + qd_p$, where q is an arbitrary polynomial. Many different choices:

- Minimal numerator degree $\deg n_c < \deg d_p$, generically
  $\deg d_c = \deg n_c = \deg d_p - 1,\ \deg d_{cl} = 2\deg d_p - 1$
  (Luenberger observer)
  $\deg d_c = \deg d_p, \quad \deg n_c = \deg d_p - 1, \deg d_{cl} = 2\deg d_p$
  (Kalman filter)
- Minimal denominator degree $\deg d_c \leq \deg n_p$ (controller has excess of zeros, derivative action). Generically
  $\deg d_c = \deg n_p - 1,\ \deg n_c = \deg d_p - 1,\ \deg d_{cl} =$
  $\deg d_p + \deg n_p - 1$
  $\deg n_p = 0,\ d_c = 1,\ \deg n_c = \deg d_p - 1,\ \deg d_{cl} = \deg d_p$
- Integral action: Add s as an extra factor of $d_p(s)$ solve for d_c and n_c and the controller is then $C = n_c(s)/(sd_c(s))$.

Non-uniqueness II

Process and controller transfer functions

$$P(s) = \frac{n_p(s)}{d_p(s)}, \qquad C(s) = \frac{n_c(s)}{d_c(s)}$$

Closed loop characteristic equation

$$d_p(s)d_c(s) + n_p(s)n_c(s) = d_{cl}(s)$$

If $C_0 = n_{c0}(s)/d_{c0}(s)$ is a controller that gives the closed loop characteristic polynomial $d_{cl}(s)$ then the controller

$$C(s) = \frac{n_{c0}(s) + q(s)d_p(s)}{d_{c0}(s) - q(s)n_p(s)}$$

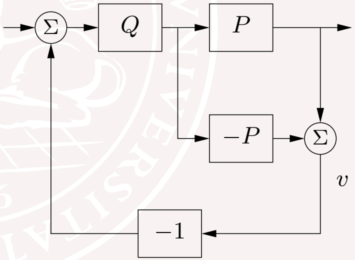
where $q(s)$ is an arbitrary polynomial also gives char. pol $d_{cl}(s)$.

$$d_p(s)\big(d_{c0}0(s) - q(s)n_p(s)\big) + n_p(s)\big(n_{c0} + q(s)d_p(s)\big) =$$
$$d_p(s)d_{c0}(s) + n_p(s)n_{c0}(s) = d_{cl}(s)$$

Youla-Kucera Parametrization I

Consider a process with a *stable* transfer function P. Let the desired transfer function from reference to output be T . Let the requirement be realized by feedforward with the transfer function Q, where $T = PQ$. Since Q must be stable, T and P should have the same zeros in the right half plane. The transfer function T can also be obtained by error feedback with the controller

$$C = \frac{Q}{1 - PQ}$$



For any stable Q, GoF is linear in Q:

$$T = PQ, \quad S = 1 - T = 1 - PQ$$
$$PS = P(1 - PQ), \quad CS = Q$$

All stabilizing controllers C can be represented by some such Q.

Youla-Kučera Parametrization II

Process transfer function $P = B/A$, where A and B are stable co-prime rational functions. Assume that the controller $C_0 = G_0/F_0$ stabilizes P. All stabilizing controllers are given by

$$C = \frac{G_0 + QA}{F_0 - QB}$$

Q is an arbitrary stable rational transfer function. GoF:

$$S = \frac{A(F_0 - QB)}{AF_0 + BG_0}, \qquad PS = \frac{B(F_0 - QB)}{AF_0 + BG_0}$$

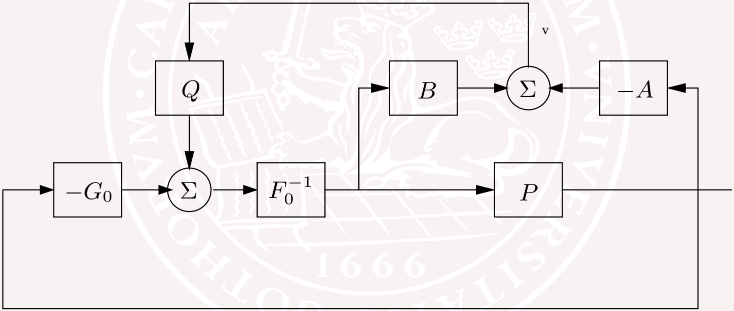
$$CS = \frac{A(G_0 + QA)}{AF_0 + BG_0}, \qquad T = \frac{B(G_0 + QA)}{AF_0 + BG_0}$$

The system is stable since the rational function $AF_0 + BG_0$ has all its zeros in the left half plane and A, B, F_0, G_0 and Q are stable rational functions.

Block Diagram Interpretation

Controller

$$C = \frac{G_0 + QA}{F_0 - QB}, \qquad F_0 U = -G_0 Y + Q(BU - AY)$$



Notice that the input to Q is nominally zero

Where to Place the Poles - Standard Forms

Butterworth configurations

$$G(s) = \frac{\omega_0^2}{s^2 + 1.414\omega_0 s + \omega_0^2}, \qquad G(s) = \frac{\omega_0^3}{s^3 + 2\omega_0 s^2 + 2\omega_0^2 s + \omega_0^3}$$

$$G_1(s) = \frac{a_n \omega_0^n}{s^n + a_1\omega_0 s^{n-1} + a_2\omega_0^2 s^{n-2} + \cdots + a_n\omega_0^n}$$

$$G_2(s) = \frac{a_{n-1}\omega_0^{n-1} s + a_n\omega_0^n}{s^n + a_1\omega_0 s^{n-1} + a_2\omega_0^2 s^{n-2} + \cdots + a_n\omega_0^n}$$

G_1 zero step error min ITAE, G_2 zero ramp error min ITAE

	$G_1(s)$					$G_2(s)$				
n	a_1	a_2	a_3	a_4	a_5	a_1	a_2	a_3	a_4	a_5
1	1					1				
2	1.505	1				3.2	1			
3	1.783	2.721	1			1.75	3.25	1		
4	1.953	3.247	2.648	1		2.41	4.93	5.14	1	
5	2.068	4.499	4.675	3.252	1	2.19	6.50	6.30	5.24	1

http://www.mathworks.com/matlabcentral/fileexchange/18547-the-optimal-itae-

Summary - Pole Placement

- Simple useful method easy to use for low order systems
- The only requirement is that there are no common zeros in the numerator and denominator polynomials of the process transfer function, or equivalently that the system is reachable and observable.
- There are multivariable extensions based on matrix polynomials
- BUT polynomial computations are numerically poorly conditioned for high order systems

$$s^n = \epsilon, \quad s = \epsilon^{1/n}, \qquad n = 10, \epsilon = 10^{-8}, \quad \epsilon^{1/n} = 0.16$$

- State space approach gives more reliable computations and other insights

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To Add

- Integral action
- Interpretation 1: dynamics of a controller arises from the observer
- Nice interpretation of closed loop characteristic polynomial
  $\det(sI - A + BL) \times \det(sI - A + KC)$
- Emphasize different 2DOF architectures
- Can we separate a given polynomial?
- How to apply the design rules - where should cancellation take part L or K?

Combining State Feedback with an Observer

Consider a system which is assumed to be observable and reachable

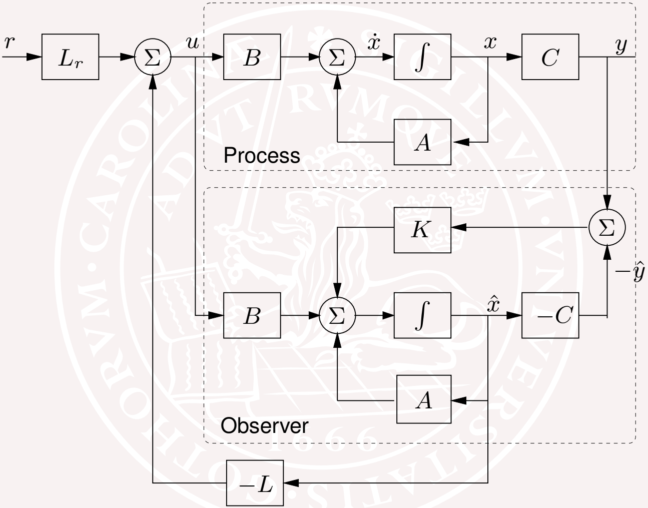
$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx$$

Determine the state with an observer and use state feedback from the observed state. This gives the controller

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$
$$u = -L\hat{x} + L_r r$$

The controller is a dynamical system whose dynamics is represented by the observer dynamics (the internal model principle). Controller complexity is given by model complexity.

Architecture of Basic Controller



The controller contains a model of the process and its environment

The Closed Loop System 1

Process

$$\frac{dx}{dt} = Ax + Bu, \qquad y = Cx.$$

Controller

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x}), \qquad u = -L\hat{x} + L_r r.$$

Introduce the state $\tilde{x} = x - \hat{x}$ instead of $\hat{x}$.

$$\begin{aligned}\frac{dx}{dt} &= Ax + Bu = Ax - BL\hat{x} + Bl_r r\\ &= (A - BL)x + BL\tilde{x} + Bl_r r\\ \frac{d\tilde{x}}{dt} &= (A - KC)\tilde{x}\end{aligned}$$

If the system is observable the observer can be designed so that the observer error $\tilde{x}$ goes to zero as fast as desired. The system behaves like a system with state feedback when $\tilde{x} = 0$

Closed Loop System 2

The closed loop system

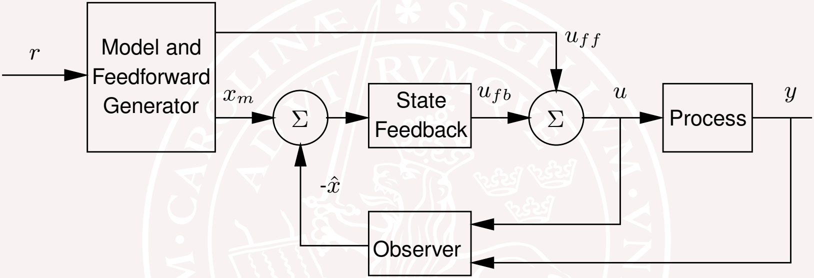
$$\frac{d}{dt}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} A - BL & BL \\ 0 & A - KC \end{pmatrix}\begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} Bl_r \\ 0 \end{pmatrix} r$$

$$y = \begin{pmatrix} C & 0 \end{pmatrix}$$

Observe

- Block triangular structure of dynamics matrix
- Closed loop eigenvalues are eigenvalues of $A - BL$ and $A - KC$
- State feedback gain L and observer gain K can be designed independently
- The state $\tilde{x}$ is not reachable from the reference. Intuitively: changes in the reference should not generate observer errors!

More Flexible Architecture with Full 2DOF



A nice two degree-of-freedom (2DOF) structure which decouples response to disturbances (handled by state feedback and observer) from response to reference signals (handled by reference model and feedforward)

There are many ways to generate the signals x_m and u_{ff}

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Robustness

- Intuitively we may expect that well-damped closed loop poles would guarantee robustness
- Unfortunately this is not true!!!
- Always necessary to check robustness if it is not part of design process
- Always check requirements that are not explicit requirements in the design procedure, particularly if you optimize
- Looking at the Gang of Four is a good idea
- A long forgotten problem (Mats Lilja's PhD #31 1989)
- Two examples that give insight
- Two simple design rules for placing poles properly

Example 1

Consider a first order system with PI control

$$P(s) = \frac{b}{s+a} = \frac{1}{s+1}, \quad C(s) = k_p + \frac{k_i}{s}$$

where the controller parameters are chosen to give a closed loop system with the characteristic polynomial $s^2 + \omega_0 s + \omega_0^2$.

Characteristic polynomial

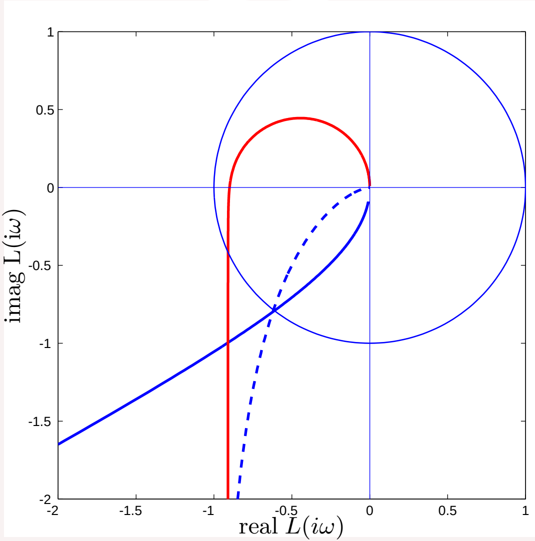
$$s(s+a) + b(k_p s + k_i) = s^2 + \omega_0 s + \omega_0^2$$

Controller parameters

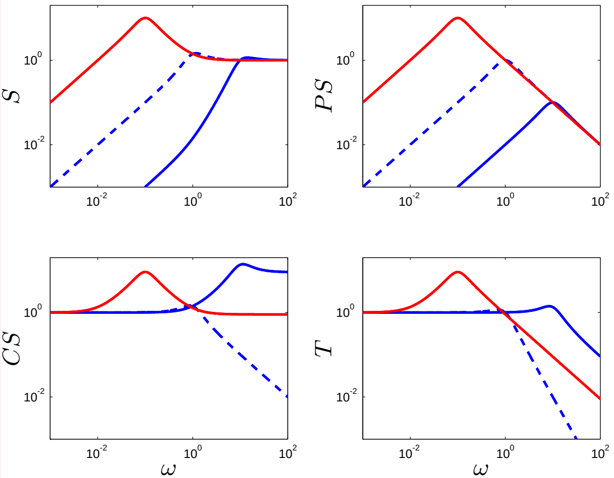
$$k_p = \frac{\omega_0 - a}{b} = \omega_0 - 1, \qquad k_i = \frac{\omega_0^2}{b} = \omega_0^2$$

What is special about $\omega_0 = 1$? What does it mean that k_p is negative?

Nyquist Plot $\omega_0/a = 0.1$ (red), 1 and 10 (blue)



Gang of Four $\omega_0/a = 0.1$ (red), 1 and 10 (blue)



Looks OK for $\omega_0/a = 1$ and 10 BUT not for $\omega_0 = 0.1$ (red curves)

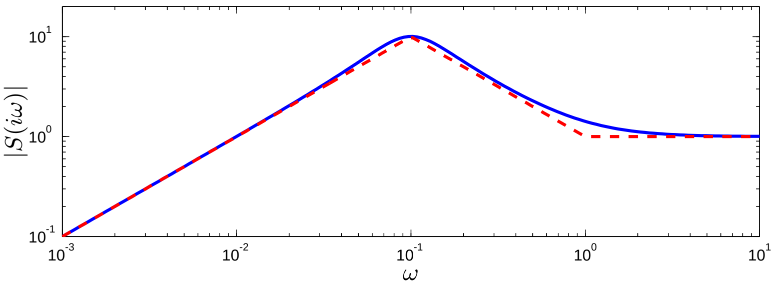
Reflections

- We have made what looks like a perfectly reasonable pole placement design with nicely damped closed loop poles $\zeta_0 = 0.5$.
- The results look good for $\omega_0/a = 1$ and 10
- The design for $\omega_0/a = 0.1$ have very high sensitivities $M_s = 9.4$ and $M_t = 10$
- It is apparently important where we place the poles
- Can we understand what goes on and fix it?

The Sensitivity Function

We have for $a = 1$ and $\omega_0 = 0.1$, $M_s \approx \dfrac{0.1}{0.011} = 9$ (9.4)

$$S = \frac{(s+a)s}{s^2+\omega_0 s+\omega_0^2} = \frac{(s+1)s}{s^2+0.1s+0.01} = \frac{d_p(s)d_c(s)}{d_p(s)d_c(s)+n_p(s)n_c(s)}$$



What creates the peak? Start from high frequencies.

Generalization

Transfer functions of process and controller

$$P(s) = \frac{n_p(s)}{d_p(s)}, \qquad C(s) = \frac{n_c(s)}{d_c(s)},$$

Sensitivity functions

$$S(s) = \frac{1}{1+PC} = \frac{d_p(s)d_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

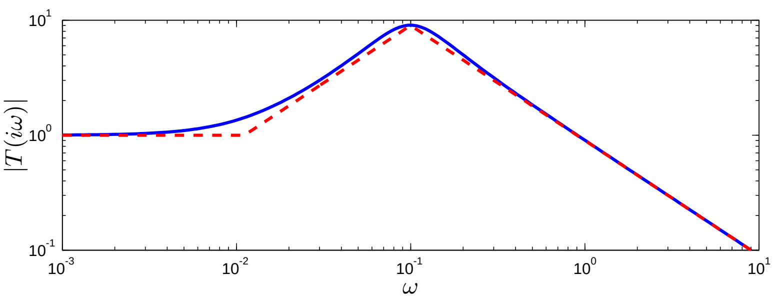
At high frequencies we have $S \approx 1$. As the frequency decreases there will be a break-point at the process poles (zeros of d_p. To avoid having high sensitivities high frequency process poles must be matched by corresponding closed loop poles. In the example there was a process pole at $s = 1$ but the closed loop poles were at 0.1.

Complementary Sensitivity Function

We have for $a = 1$ and $\omega_0 = 0.1$

$$T = \frac{(\omega_0 - 1)s + \omega_0^2}{s^2 + \omega_0 s + \omega_0^2} = \frac{-0.9s + 0.01}{s^2 + 0.1s + 0.01} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

Notice slow zero in the controller!!



We have approximately $M_t \approx \frac{0.1}{0.01} = 10$ (10.04)

Example 2

Consider the process and controller

$$P(s) = \frac{b_1 s + b_2}{s^2}, \qquad C(s) = \frac{g_0 s + g_1}{s^2 + f_1 s + f_2}$$

Desired closed loop characteristic polynomial

$$d_{cl}(s) = (s^2 + 2\zeta_c\omega_c s + \omega_c^2)(s^2 + 2\zeta_o\omega_o s + \omega_o^2)$$

We have

$$s^2(s^2 + f_1 s + f_2) + (b_1 s + b_2)(g_0 s + g_1) = c(s)$$

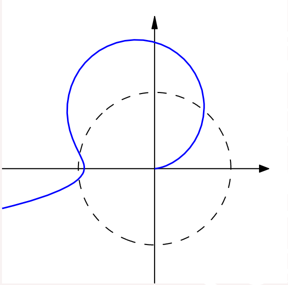
Identification of coefficients of equal powers of s gives

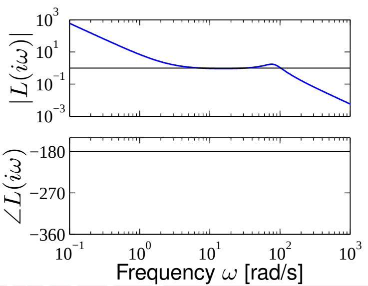
$$\begin{aligned}
f_1 &= 2(\zeta_o\omega_o + \zeta_c\omega_c) \\
f_2 &= \omega_o^2 + \omega_c^2 + 4\zeta_0\zeta_c\omega_o\omega_c - b_1 g_0 \\
g_0 &= \frac{2b_2(\zeta_o\omega_c + \zeta_c\omega_o)\omega_o\omega_c - b_1\omega_0^2\omega_c^2}{b_2^2} \\
g_1 &= \omega_0^2\omega_c^2 / b_2
\end{aligned}$$

Another Simple Pole Placement Problem ...

With $b_1 = 0.5$, $b_2 = 1$, $\omega_c = 10$, $\zeta_c = 0.707$, $\omega_o = 20$ and $\zeta_o = 0.707$ we get

$$C(s) = \frac{-11516s + 40000}{s^2 + 42.4s + 6658}.$$



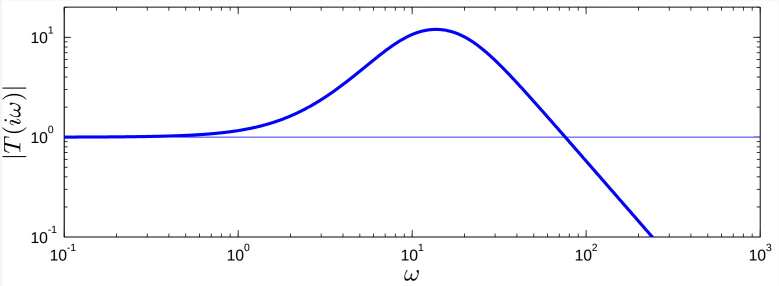


Extremely poor robustness $M_s = 13$ and $M_t = 12$

Complementary Sensitivity Function

$$P(s) = \frac{0.5s + 1}{s^2}, \qquad C(s) = \frac{-11516s + 40000}{s^2 + 42.4s + 6658}$$

Notice slow zero in the controller!!



Generalization

Transfer functions of process and controller

$$P(s) = \frac{n_p(s)}{d_p(s)}, \qquad C(s) = \frac{n_c(s)}{d_c(s)},$$

Sensitivity functions

$$T(s) = \frac{PC}{1+PC} = \frac{n_p(s)n_c(s)}{d_p(s)d_c(s) + n_p(s)n_c(s)}$$

At low frequencies we have $T \approx 1$. As the frequency increases there will be breakpoints at the process zeros of (zeros of n_p). To avoid having high sensitivities low frequency process zeros must be matched by corresponding closed loop poles. In the example there was a process zero at $s = 0.5$ but the slowest closed loop poles were at $s = 10$, hence a peak of ≈ 10.

Design Rules

- Formally only reachability and observability are required
- To obtain robust closed loop systems the poles and zeros of the process must be taken into account. Design rule.
- Choose bandwidth ω_b or dominating closed loop poles: classify poles and zeros as slow $< \omega_b$ or fast $> \omega_b$
- Slow unstable zeros (time delays) and fast unstable poles restrict the choice of closed loop bandwidth ω_b
- Design rule: *Pick closed loop poles close to slow stable process zeros and fast stable process poles.* Picking closed loop poles and zeros identical to slow stable zeros and fast stable poles give cancellations and simple calculations.
- Violating the design rule leads to closed loop systems that are not robust.
- Unstable poles and zeros cannot be canceled, slow unstable zeros and fast unstable poles therefore give fundamental limitations.

1. Introduction
2. Simple Examples
3. Polynomial Design
4. State Space Design
5. Robustness and Design Rules
6. Model Reduction
7. Oscillatory Systems
8. Summary

Theme: Be aware where you place them!

Model Reduction - Fast Stable Poles

Neglect fast modes in the model, explore a simple case

$$P(s) = \frac{a}{s+a} \approx 1$$

Reduced order model is a static system, I controller sufficient

$$C(s) = \frac{k_i}{s}$$

Choose k_i to give the closed loop pole a_0, hence $k_i = a_0$, closed loop characteristic polynomial with true plant model

$$s(s+a) + aa_0 = 0$$

$$s = -a_0 - \frac{a_0^2}{a} - \dots, \qquad s = -a + a_0 + \frac{a_0^2}{a} - \dots$$

- Simple rule: neglect stable poles that are an order of magnitude larger than a_0
- Notice a closed loop pole close to neglected pole a

Model Reduction - Slow Zeros

For large ω_0 we can approximate the process model by neglecting slow zeros

$$P(s) = \frac{s+a}{s^2} \approx \frac{1}{s}$$

Use a PI controller $k_p = 2\zeta\omega_0, k_i = \omega_0^2$.

Closed loop characteristic polynomial with true model

$$s^3 + (s+a)(k_p s + k_i) = s(s^2 + k_p s + k_i) + a(k_p s + k_i) = 0$$

- For small a roots close to $s^2 + k_p s + k_i = 0$
- For small s roots close to $s + a = 0$
- Simple rule: neglect stable zeros that an order of magnitude smaller than ω_0

Design Rules

- Simple models are useful
- How to approximate?
- Start with desired bandwidth ω_{bw} which is of the order of $\omega_{ms}, \omega_{mt}, \omega_{gc}$
- Neglect process poles and zeros that are an order of magnitude faster than ω_{bw} unless they are highly oscillatory.
- Neglect process zeros that are an order of magnitude slower than ω_{bw}, approximate slow poles by integrators.
- Cancel fast process poles and slow process zeros that cannot be neglected. Make the design based on the simplified model, add the canceled factors to the controller after the design. Add high frequency roll-off if necessary.

An Example

Consider

$$P(s) = \frac{K_p(1+sT_4)(1+sT_5)}{(1+sT_1)(1+sT_2)(1+sT_3)} e^{-sL}$$

where T_1 is significantly larger than L and the other time constants.

For controller with very low performance, settling times slower than T_1, approximate the process by its gain and use an integrating controller.

For slightly higher performance approximate by

$$P(s) \approx \frac{K_p}{1+s(T_1+T_e)} e^{-sT_e} \quad \text{or} \quad P(s) \approx \frac{K_p(1-sT_e)}{(1+sT_1)(1+sT_e)}$$

where $T_e = (T_2 + T_3 + L - T_4 - T_5)/2$ and $T_e > 0$

Pole Placement Design

Theme: Be aware where you place them!

Oscillatory Systems

Truxal 1961: The design of feedback systems to effect satisfactorily the control of very lightly damped physical systems is perhaps the most basic of the difficult control problems.

A simple prototype problem

$$P(s) = \frac{\omega_0^2}{s^2 + 2\zeta_0\omega_0 s + \omega_0^2}$$

with very low damping $\zeta = 0.01$ ($Q = 50$)

- Integral (I) control
- PI control
- PID Control - Notch-filter design
- PID Control - Active damping of oscillatory modes
- Systems with many resonances

Integrating (I) Control

Any stable system with $P(0) \neq 0$ can be controlled by an integrating controller provided that requirements are modest. Approximate the process transfer function by neglecting the oscillatory mode

$$P(s) = \frac{\omega_0^2}{s^2 + 2\zeta_0\omega_0 s + \omega_0^2} \approx 1$$

With pure integral control we have

$$C(s) = \frac{k_i}{s}, \qquad L(s) = \frac{k_i\omega_0^2}{s(s^2 + 2\zeta_0\omega_0 s + \omega_0^2)} \approx \frac{k_i}{s}$$

Requirements on the gain margin limits integral gain $k_i = \omega_{gc}$

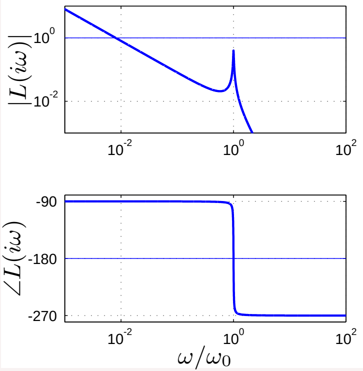
$$g_m = -\frac{1}{L(i\,\omega_0)} = \frac{2\zeta_0\omega_0}{k_i}, \qquad k_i = \frac{2\zeta_0\omega_0}{g_m}, \qquad \omega_{gc} = \frac{\omega_0}{Qg_m}$$

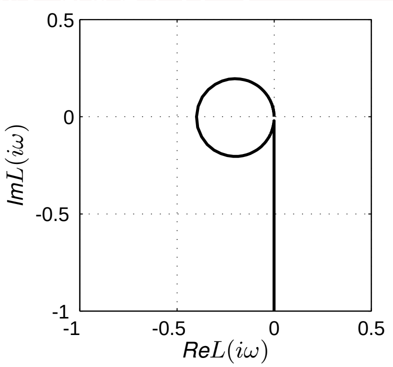
I Control

Any stable system with $P(0) \neq 0$ can be controlled by an integrating controller provided that requirements are modest.

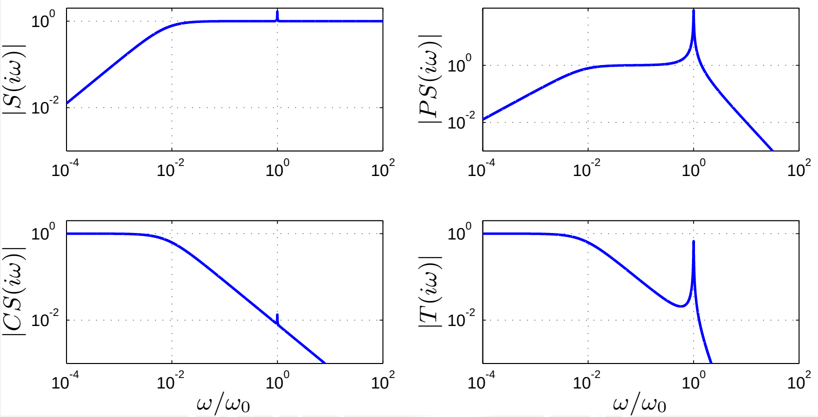
$$L(s) = \frac{k_i \omega_0^2}{s(s^2 + 2\zeta_0\omega_0 s + \omega_0^2)}, \qquad k_i = \frac{2\zeta_0\omega_0}{g_m}$$

Numerical values: $\zeta_0 = 0.01$, $g_m = 2.5$ ($\Rightarrow M_s \approx 1.7$) gives $k_i = 0.008\omega_0$





Gang of Four for Integral Control



- Good robustness $M_s = 1.7$, $M_t = 1$
- low bandwidth $\omega_{gc} = 0.008\omega_0$
- extra sensitive to load disturbances at resonance frequency

I Control with Roll-off

Controller transfer function with high-frequency roll-off

$$C(s) = \frac{k_i}{s(1 + sT_f + s^2T_f^2/2)}$$

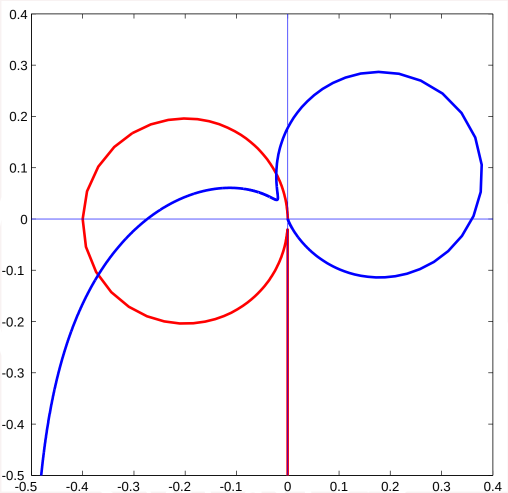
Loop transfer function and low frequency approximation

$$L(s) = \frac{k_i\omega_0^2}{s(s^2T_f^2/2 + sT_f + 1)(s^2 + 2\zeta_0\omega_0 s + \omega_0^2)}$$

$$\approx \frac{k_i}{s}(1 - sT_f - s\frac{2\zeta_0}{\omega_0}) \approx \frac{k_i}{s} - k_iT_f$$

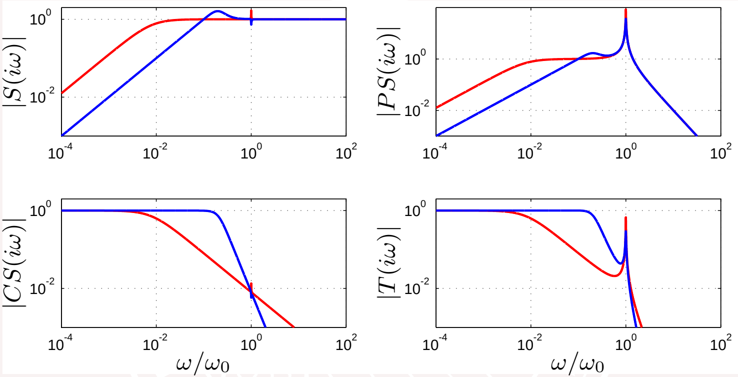
Choosing $T_f = 0.5/k_i$ gives a Nyquist curve that is close to the robustness valley ($\mathrm{Re}\,\mathrm{L}(\mathrm{i}\omega) = -0.5$) for small s, adjust k_i to be as large as possible without sacrificing robustness, $k_i = 0.1\omega_0$ an order of magnitude larger than without a filter ($k_i = 0.008\omega_0$).

I Control with Roll-off ... Nyquist Plot



- The filter shifts the red circle to a safer region (2nd order filter important!)

I Control with Roll-off ...



- I Control: $k_i = 0.008\,\omega_0$ (red)
- I Control: with filter: $k_i = 0.1\,\omega_0$ (blue)
- The filter gives a significant improvement at least 10X

Summary of I Control

- Gain crossover frequency is limited to

$$k_i = \omega_{gc} < \frac{2\zeta_0\omega_0}{g_m} = 0.008\,\omega_0$$

- Proportional action does not give any improvements
- Roll-off filter helps to increase k_i significantly to $0.1\omega_0$
- Notice bending of the loop to a safer region
- High resonant peak in PS remains
- Controller noise gains CS moderate

PID Control - Notch filter design

The resonant poles must be considered in order to increase the bandwidth. If the bandwidth is less than ω_0 we must choose closed loop poles that are close to the process poles. A simple way to do this is to cancel the fast process poles by controller zeros. Controller ($C(s)$) and the loop transfer function ($L(s)$) become

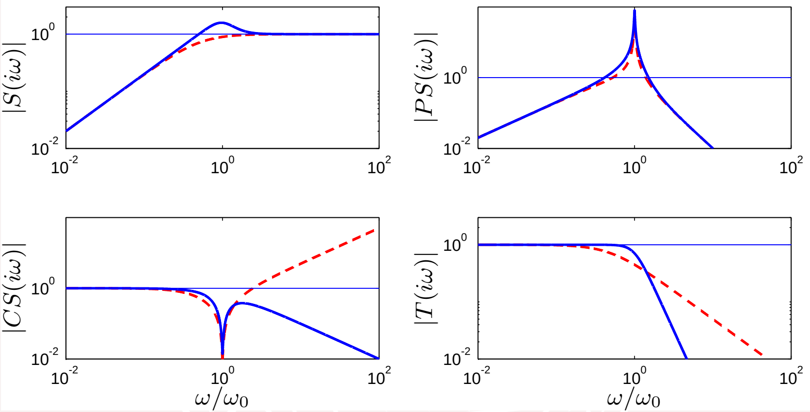
$$C(s) = k_i \frac{s^2 + 2\zeta_0\omega_0 s + \omega_0^2}{\omega_0^2 s} = \frac{k_d s^2 + k_p s + k_i}{s}, \qquad L(s) = \frac{k_i}{s}$$

Add *high frequency roll-off*

$$C(s) = \frac{k_i(s^2 + 2\zeta_0\omega_0 s + \omega_0^2)}{\omega_0^2 s(1 + sT_f + s^2T_f^2/2)}, \quad L(s) = \frac{k_i}{s(1 + sT_f + s^2T_f^2/2)}$$

Relate T_f to gain crossover frequency $\omega_{gc} \approx k_i, T_f k_i = 0.5$

Gang of Four for PID Notch Filter Control



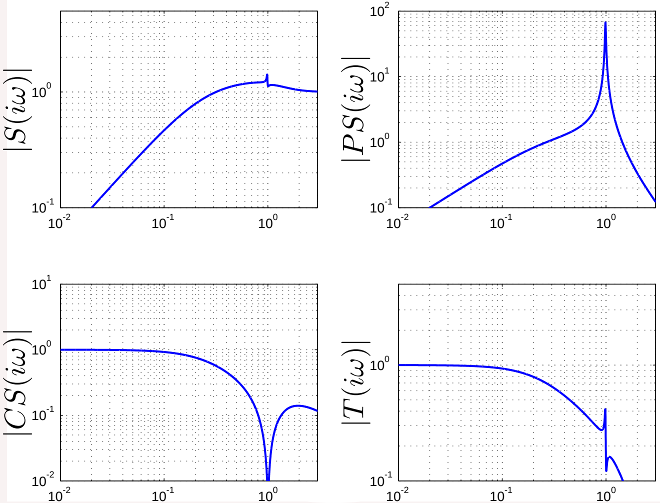
$k_i = 0.5\omega_0$, $T_f = 0.5/k_i$, $M_s = 1.6$, $M_t = 1$ (robustness valley!)

Without filter red, with filter blue

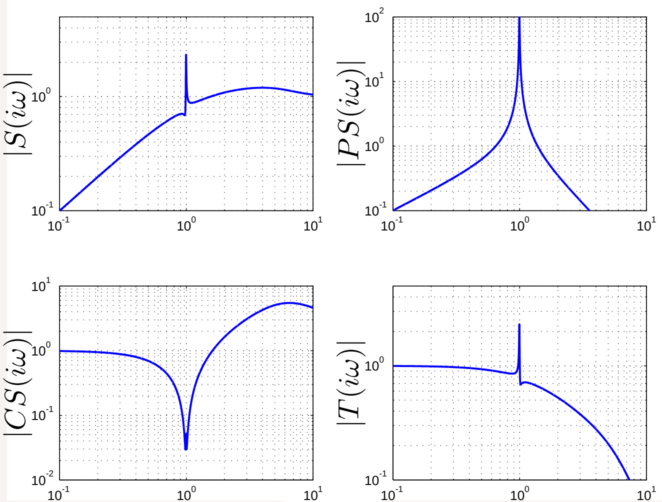
High peak in response to load disturbances.

High frequency roll-off important

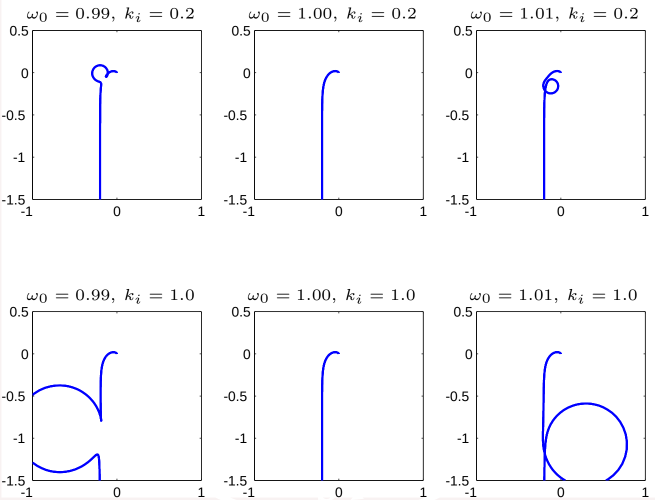
Parameter Variations 1% for $k_i = 0.2\,\omega_0$



Parameter Variations 1% for $k_i = \omega_0$



Nyquist Plots for $k_i = 0.2\,\omega_0$ and $k_i = \omega_0$



Underestimating ω_0 better than overestimating ($\omega_{design} = 1.00$)

Summary of Notch-filter Design

- Bandwidth can be increased significantly compared to I-Control
- Maximum sensitivities are OK.
- Small parameter variations give moderate changes for designs with $\omega_b = k_i = 0.2\omega_0$ but very high sensitivities for designs with $k_i = \omega_0$.
- High sensitivity to small time delay variations for designs with $k_i = \omega_0$.
- Still very sensitive to disturbances with energy at the resonance frequency, which are not attenuated by the feedback loop.

Active Damping of Resonant Mode

A PID controller is sufficient. The loop transfer function is

$$L(s) = \frac{(k_d s^2 + k_p s + k_i)\omega_0^2}{s(s^2 + 2\zeta_0\omega_0 s + \omega_0^2)}.$$

Closed loop characteristic polynomial

$$s^3 + (k_d\omega_0^2 + 2\zeta_0\omega_0)s^2 + (k_p + 1)\omega_0^2 s + k_i\omega_0^2.$$

A general third order polynomial can be parameterized as

$$\begin{aligned}(s + \alpha_c\omega_c)&(s^2 + 2\zeta_c\omega_c s + \omega_c^2)\\ &= s^3 + (\alpha_c + 2\zeta_c)\omega_c s^2 + (1 + 2\alpha_c\zeta_c)\omega_c^2 s + \alpha_c\omega_c^3.\end{aligned}$$

Identification of coefficients of equal powers of s give

$$k_d = \frac{(\alpha_c + 2\zeta_c)\omega_c - 2\zeta_0\omega_0}{\omega_0^2}, \quad k_p = \frac{(1 + 2\alpha_c\zeta_c)\omega_c^2}{\omega_0^2} - 1, \quad k_i = \frac{\alpha_c\omega_c^3}{\omega_0^2}$$

The Controller

Add high frequency roll-off

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s(1 + sT_f + s^2 T_f^2/2)}$$

Parameters:

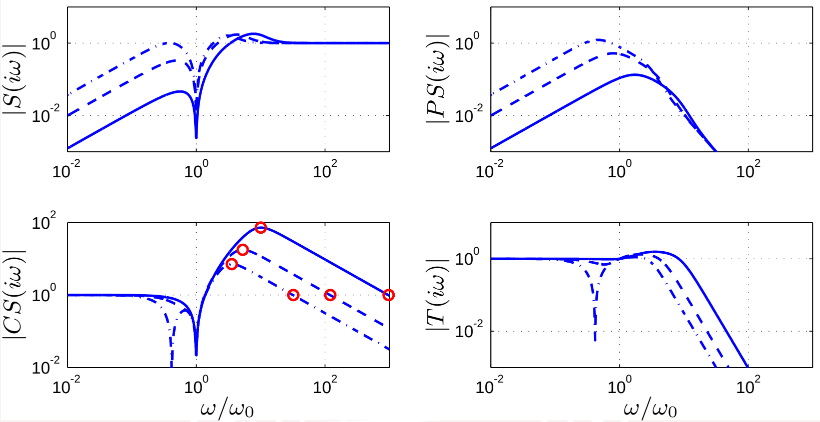
$$k_d = \frac{(\alpha_c + 2\zeta_c)\omega_c - 2\zeta_0\omega_0}{\omega_0^2}$$

$$k_p = \frac{(1 + 2\alpha_c\zeta_c)\omega_c^2}{\omega_0^2} - 1$$

$$k_i = \frac{\alpha_c\omega_c^3}{\omega_0^2}$$

$$T_f = \frac{T_d}{10} = \frac{k_d}{10k_p}$$

Changing Integral Gain $k_i = 0.27\omega_0, \omega_0, 8\,\omega_0$



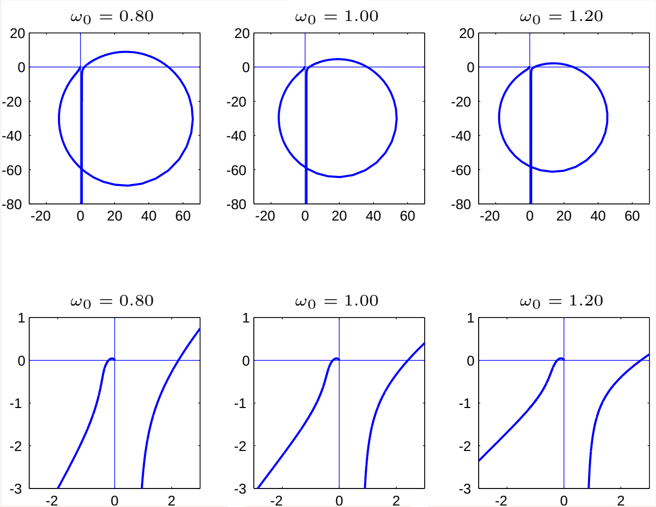
$k_i = 0.27\omega_0$ (dash-dotted), $k_i = \omega_0$ (dashed) and $k_i = 8\omega_0$, (full).
Notice dramatic increase of bandwidth and controller HF gain and bandwidth!
Bandwidth increased while maintaining robustness $M_s = 1.7$, $M_t = 1.6$.
But we have to look at parametric uncertainty too!

Admissible High Frequency Controller Gain

To estimate allowable high frequency gains we must consider

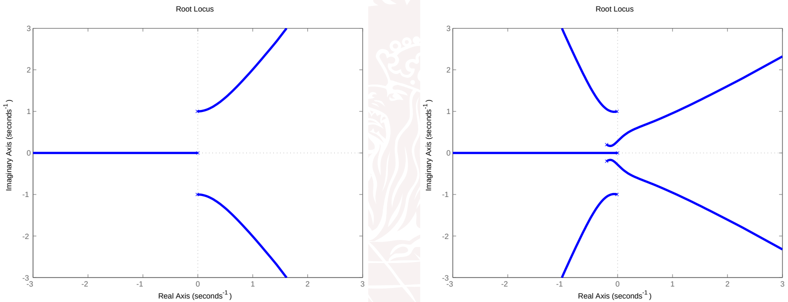
- High frequency measurement noise
- Range of control signal
- Bandwidth of sensors and actuators
- Model accuracy
- Robustness

Robustness to Parameter Variations

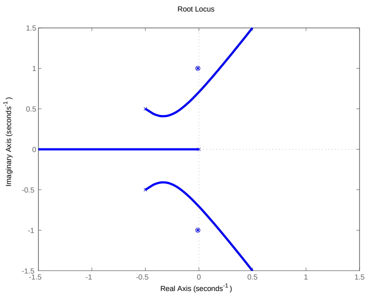


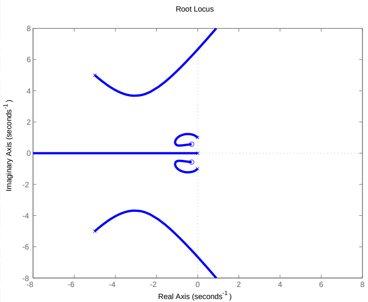
Not very sensitive to parameter variations

Closed Loop Poles I and Ifilt



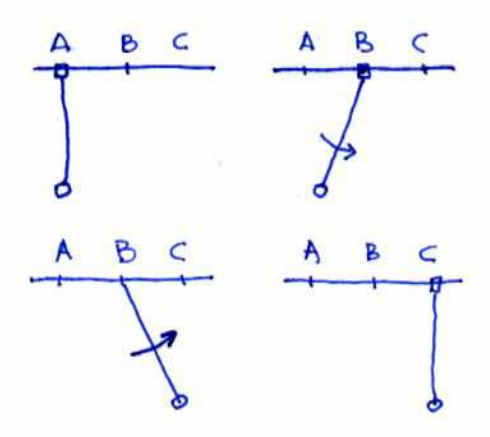
Closed Loop Poles Notch and Damp





Posicast Control

- A scheme for command signal control of oscillatory systems
- Proposed by Otto Smith who also invented the Smith predictor
- The idea: move a hanging load from one position to another

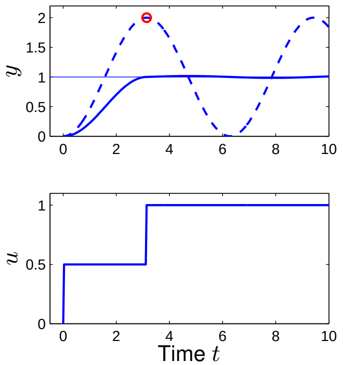


Time Response and Transfer Function

Transfer function interpretation

$$G(s) = \frac{1}{2}(1 + e^{-sT_p/2})$$

Notice that $G(i\omega) = 0$ for $\omega T_p = 2\pi + 4n\pi$



- Posicast controller is an efficient notch filter
- Can be modified to handle damped oscillations
- Application to MEMS drives

I Control with I Posicast Notch with Filter

Controller transfer function

$$C(s) = \frac{k_i}{s}\left(\gamma + (1-\gamma)e^{-sT_d}\right)$$

$$\gamma = \left(1 + e^{-\zeta_0\pi/\sqrt{1-\zeta_0^2}}\right)^{-1}, \qquad T_d = \frac{\pi}{\omega_0\sqrt{1-\zeta_0^2}}$$

Loop transfer function

$$L(s) = \frac{k_i\left(\gamma + (1-\gamma)e^{-sT_d}\right)\omega_0^2}{s(s^2 + 2\zeta_0\omega_0 s + \omega_0^2)}.$$

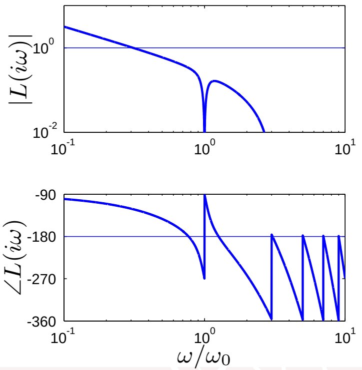
Series expansion for small s

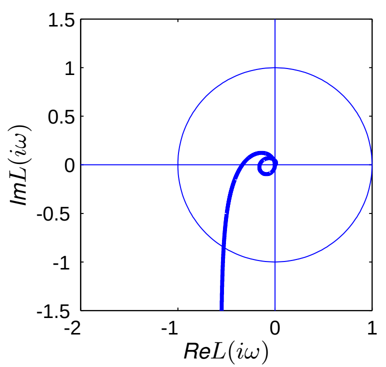
$$L(s) \approx \frac{k_i}{s}\left(1 - (1-\gamma)T_d s - \frac{2\zeta_0 s}{\omega_0}\right) = \frac{k_i}{s} - k_i\frac{(1-\gamma)\pi}{\omega_0\sqrt{1-\zeta_0^2}} - \frac{2\zeta_0 k_i}{\omega_0}$$

Pick k_i is so that $\mathrm{Re}\,\mathrm{L}(\mathrm{i}\omega) = -0.5$ for small s,

$$k_i = 0.5\frac{\omega_0\sqrt{1-\zeta_0^2}}{(1-\gamma)\pi + 2\zeta_0\sqrt{1-\zeta_0^2}}.$$

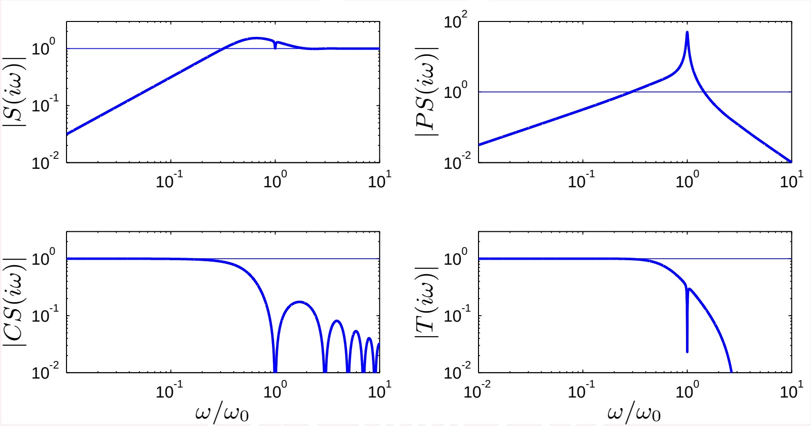
Bode and Nyquist Plots





Integral gain $k_i = 0.32\,\omega_0$, bandwidth $\omega_B = 0.63\,\omega_0$ maximum sensitivities: $M_s = 1.53$, $M_t = 1.00$, $M_{ps} = 50.0$, $M_{cs} = 1.00$.

Gang of Four



Integral gain $k_i = 0.32\,\omega_0$, bandwidth $\omega_B = 0.63\,\omega_0$ maximum sensitivities: $M_s = 1.53$, $M_t = 1.00$, $M_{ps} = 50.0$, $M_{cs} = 1.00$

Summary

- I control: $k_i \leq 2\zeta_0\omega_0/g_m$ ($k_i = 0.008\,\omega_0$), high sensitivity for disturbances at resonance frequency
- I control with filter: $k_i = 0.1\,\omega_0$, high sensitivity for disturbances at resonance frequency
- I control with posicast notch $k_i = 0.32\,\omega_0$, high sensitivity to disturbance at resonance frequency
- PI: Proportional action gives no improvement
- PID Notch with roll-off: $k_i = 0.8\,\omega_0$ high sensitivity for disturbances at resonance frequency, high sensitivity to variations in resonance frequency
- PID with active damping and roll-off: $k_i = 8\,\omega_0$, sensitivity for disturbances at resonance frequency reduced, requires high gain of controller at high frequencies $M_{cs} = 100$, and a wide band controller $\omega_{ccs} = 200\,\omega_0$

Pole Placement Design

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Summary

- Simple design method for SISO systems
- Probably quickest way to introduce design
- Most importantly we get insight and design rules
- Insights through Euclid's algorithm and Youla parametrization
- Polynomials are bad numerically, matrix calculations much more robust
- There are multivariable versions but they are complicated

Reading Suggestions

Åström Murray Feedback Systems - An Introduction for Scientists and Engineers, Princeton 2008 (also on Richards home page). GoF Ch 11. Use index for other things.

Åström Hägglund. Advanced PID control - Second edition has full chapter on oscillatory systems - drafts available